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Higher Gauge Theory: 2-Connections on 2-Bundles

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Abstract: Connections and curvings on gerbes are beginning to play a vital role in differential geometry and mathematical physics — first abelian gerbes, and more recently nonabelian gerbes. These concepts can be elegantly understood using the concept of '2-bundle' recently introduced by Bartels. A 2-bundle is a generalization of a bundle in which the fibers are categories rather than sets. Here we introduce the concept of a '2-connection' on a principal 2-bundle. We describe principal 2-bundles with connection in terms of local data, and show that under certain conditions this reduces to the cocycle data for nonabelian gerbes with connection and curving subject to a certain constraint — namely, the vanishing of the 'fake curvature', as defined by Breen and Messing. This constraint also turns out to guarantee the existence of '2-holonomies': that is, parallel transport over both curves and surfaces, fitting together to define a 2-functor from the 'path 2-groupoid' of the base space to the structure 2-group. We give a general theory of 2-holonomies and show how they are related to ordinary parallel transport on the path space of the base manifold.

Contents

1. Introduction

The gauge principle and the concept of connection is at the very heart of modern physics, and is also central to much of modern mathematics. It is all about parallel transport along *curves*. Due to the influence of string theory on the one hand (see §1.2 (p.7)) and higher category theory on the other (see §2.1 (p.9)), there are compelling reasons to generalize this concept to higher dimensions and find a notion of parallel transport along *surfaces*.

1.1 Outline of the Results

In this paper we categorify the notion of a *principal bundle with connection*, defining *principal 2-bundles with 2-connections*. We work out their description in terms of local data and show that under certain conditions this is equivalent to the cocycle description of (possibly twisted) nonabelian gerbes satisfying a certain constraint — the vanishing of the 'fake curvature'. We show that this constraint is also sufficient to guarantee the existence of *2-holonomies*, i.e., parallel transport over surfaces. We examine these 2-holonomies in detail using 2-functors into 2-groups on the one hand, and connections on path space on the other hand.

Several aspects of this have been studied before. Categorification is described in [1] and its application to groups and Lie algebras, which yields 2-groups and Lie 2-algebras, is discussed in [2, 3, 4]. The concept of 2-group is incorporated in the definition of principal 2-bundles (without connection) in [5]. Local 2-connections as 2-functors from path 2-groupoids to 2-groups were considered in [6, 7]. Connections on path space were discussed in [8, 9], and reparametrization invariance for a special case was investigated by [10]. Cocycle data for nonabelian gerbes with connection and curving were obtained originally in [11] using algebraic geometry and later by Aschieri and Jurčo [12, 13] using a nonabelian generalization of the bundle gerbes originally introduced in [14].

Here we extend this work by:

- defining the concept of a principal 2-bundle with 2-connection,
- showing that a 2-connection on a trivial principal 2-bundle has 2-holonomies defining a 2-functor into the structure 2-group when the 2-connection has vanishing 'fake curvature' (a concept already defined for nonabelian gerbes by Breen and Messing [11]),
- clarifying the relation between connections on a trivial principal bundle over the path space of a manifold and 2-connections on a trivial principal 2-bundle over the manifold itself, showing that a connection on the path space whose holonomies are invariant under arbitrary surface reparameterizations defines a 2-connection on the original manifold,
- deriving the local 'gluing data' that describe how a nontrivial 2-bundle with 2-connection can be built from trivial 2-bundles with 2-connection on open sets that cover the base manifold,

- demonstrating that these gluing data for 2-bundles with 2-connection coincide with the cocycle description of (possibly twisted) nonabelian gerbes, subject to the constraint of vanishing fake curvature.

The starting point for all these considerations is the ordinary concept of a principal fiber bundle. Such a bundle can be specified using the following 'gluing data':

- a base manifold B,
- a cover of B by open sets $\{U_i\}_{i\in I}$,
- a Lie group G (the 'gauge group' or 'structure group'),
- on each double overlap $U_{ij} = U_i \cap U_j$ a G-valued function g_{ij},
- such that on triple overlaps the following transition law holds:

$$g_{ij} g_{jk} = g_{ik}.$$

Such a bundle is augmented with a connection by specifying:

- in each open set U_i a smooth functor $\mathrm{hol}_i \colon \mathcal{P}_1(U_i) \to G$ from the path groupoid of U_i to the gauge group,
- such that for all paths γ in double overlaps U_{ij} the following transition law holds:

$$\mathrm{hol}_i(\gamma) = g_{ij}\,\mathrm{hol}_j(\gamma)\; g_{ij}^{-1}.$$

Here the 'path groupoid' $\mathcal{P}_1(X)$ of a manifold X has points of X as objects and certain equivalence classes of smooth paths in X as morphisms. There are various ways to work out the technical details; see [15] for the approach we adopt, which uses 'thin homotopy classes' of smooth paths. Technical details aside, the basic idea is that a connection on a trivial G-bundle over X gives a well-behaved map assigning to each path γ the holonomy $\mathrm{hol}(\gamma) \in G$ of the connection along that path. Saying this map is a 'smooth functor' means that these holonomies compose when we compose paths, and that the holonomy $\mathrm{hol}(\gamma)$ depends smoothly on the path γ.

Our task shall be to categorify all of this and to work out the consequences. The basic tool will be *internalization*: given a mathematical concept X defined solely in terms of sets, functions and commutative diagrams involving these, and given any category C, one obtains the concept of an 'X in C' by replacing all these sets, functions and commutative diagrams by corresponding objects, morphisms, and commutative diagrams in C.

For example, take X to be the concept of 'group'. A group in Diff (the category with finite-dimensional smooth spaces as objects and smooth maps as morphisms) is nothing but a *Lie group*. In other words, a Lie group is a group that is a manifold, for which all the group operations are smooth maps. Similarly, a group in Top (the category with topological spaces as objects and continuous maps as morphisms) is a *topological group*.

These examples are standard, but we will need some slightly less familiar ones. In particular, we will need the concept of 'strict 2-group', which is a group in Cat (the category with categories as objects and functors as morphisms). By a charming principle called 'commutativity of internalization', strict 2-groups can also be thought of as categories in Grp (the category with groups as objects and homomorphisms as morphisms). We will also need the concept of a '2-space', which is a category either in Diff, or perhaps in some more general category of smooth spaces, allowing for infinite-dimensional examples. Finally, combining the concepts of strict 2-group and 2-space, we obtain that of 'strict Lie 2-group'. This is a strict 2-group in Diff, or equivalently, a group in the category of 2-spaces.

To arrive at the definition of a 2-bundle $P \to M$, the first steps are to replace the total space P and base space M by 2-spaces, and to replace the structure group by a strict Lie 2-group. Often it is interesting to keep the base space an ordinary space, which can be regarded as a 2-space with only identity morphisms. However, for applications to string theory we will also be interested in more general 2-bundles, where the base 2-space has some manifold X as its space of objects and the free loop space LX as its space of morphisms [16]. This sort of application also requires that we consider smooth spaces that are more general than finite-dimensional smooth manifolds.

Just as a connection on a trivial principal bundle over M gives a functor hol from the path groupoid of M to the structure group, one might hope that a 2-connection on a trivial principal 2-bundle would define a 2-functor from some sort of 'path 2-groupoid' to the structure 2-group. This has already been studied in [6, 7] in the context of lattice 2-gauge theory. Thus, the main issues not yet addressed are those involving differentiability.

To address these issues, we define for any smooth space M a smooth 2-groupoid $\mathcal{P}_2(M)$ such that:

- the objects of $\mathcal{P}_2(M)$ are points of M: $\bullet\, x$
- the morphisms of $\mathcal{P}_2(M)$ are smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(t)$ is constant in a neighborhood of $t = 0$ and $t = 1$: $\bullet \xrightarrow{\gamma} \bullet$
- the 2-morphisms of $\mathcal{P}_2(M)$ are thin homotopy classes of smooth maps $f\colon [0,1]^2 \to M$ such that $f(s,t)$ is independent of s in a neighborhood of $s = 0$ and $s = 1$, and constant in a neighborhood of $t = 0$ and $t = 1$: $\begin{tikzcd} \bullet \arrow[r, bend left=50, "\gamma_1"{name=U}] \arrow[r, bend right=50, "\gamma_2"'{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "f"] \end{tikzcd}$

We call the 2-morphisms in $\mathcal{P}_2(M)$ 'bigons'. The 'thin homotopy' equivalence relation guarantees that two maps differing only by a reparametrization define the same bigon. This is important because we seek a *reparametrization-invariant notion of surface holonomy.*

We show that any 2-connection on a principal 2-bundle over M with vanishing fake curvature yields a smooth 2-functor $\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$, where $\mathcal{G}$ is the structure 2-group. We call this 2-functor the *2-holonomy* of the 2-connection. In simple terms, its existence means the 2-connection has well-defined holonomies both for paths and surfaces, compatible with

the standard operations of composing paths and surfaces, and depending smoothly on the path or surface in question.

To expand on this slightly, one must recall [2] that any strict Lie 2-group $\mathcal{G}$ determines a 'crossed module' of Lie groups (G, H, t, α), where:

- G is the Lie group of objects of $\mathcal{G}$,
- H is the Lie group of morphisms of $\mathcal{G}$ with source equal to $1 \in G$,
- $t\colon H \to G$ is the homomorphism sending each morphism in H to its target,
- α is the action of G as automorphisms of H defined using conjugation in H as follows: $\alpha(g)h = 1_g h {1_g}^{-1}$.

In these terms, a 2-connection on a trivial principal 2-bundle over M with structure 2-group $\mathcal{G}$ is nothing but a Lie(G)-valued 1-form A together with a Lie(H)-valued 2-form B on M. Translated into this framework, Breen and Messing's 'fake curvature' is the Lie(G)-valued 2-form

$$dt(B) + F_A,$$

where $F_A = dA + A \wedge A$ is the usual curvature of A. We show that *when the fake curvature vanishes*, but not in general otherwise, there is a well-defined 2-holonomy $\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$. The importance of vanishing fake curvature in the framework of lattice gauge theory was already emphasized in [7]. The special case where also $F_A = 0$ was studied in [8], while a discussion of this constraint in terms of loop space in the case $G = H$ was given in [9]. Our result subsumes these cases in a common framework.

This framework for 2-connections on trivial 2-bundles is sufficient for local considerations. Thus, all that remains is to turn it into a global notion by categorifying the *transition laws* for a principal bundle with connection, which in terms of local data read:

$$\begin{aligned} g_{ij} g_{jk} &= g_{ik} \\ \mathrm{hol}_i(\gamma) &= g_{ij} \mathrm{hol}_j(\gamma)\, g_{ij}^{-1} . \end{aligned}$$

The basic idea is to replace these equations by specified isomorphisms, using the fact that a 2-group $\mathcal{G}$ has not only objects (forming the group G) but also morphisms (which can be describing using elements of the group H). These isomorphisms should in turn satisfy certain coherence laws of their own. These coherence laws have already been worked out for 2-bundles without connection [5] and for twisted nonabelian gerbes with connection and curving [11, 12, 13]; here we put these ideas together. We do this not for the most general 2-bundle with 2-connection, but for the special case where:

- the structure 2-group is a strict Lie 2-group (indeed, we have not even mentioned the more general 'coherent' Lie 2-groups),
- the base 2-space is either one with only identity morphisms (that is, an ordinary space) or one where the morphisms are loops,

- the arrow-part of the categorified g_{ij} is either trivial or nontrivial on 'infinitesimal' loops (where the latter are really antisymmetric rank $(2,0)$ tensors at a given base point).

The reason for these restriction is that, as we show, the local data describing such 2-bundles with 2-connections is equivalent to the cocycle data describing possibly twisted nonabelian bundle gerbes with connection and curving, subject to the constraint of vanishing fake curvature.

More precisely, in terms of the notation in [12, 13] we find that:

- the point part of g_{ij} is the gerbe transition function φ_{ij},
- the arrow part of g_{ij} defines a 2-form which transforms as the gerbe's d_{ij},
- the natural transformation in the transition law for g_{ij} encodes the gerbe 0-forms f_{ijk},
- a natural transformation between the restriction maps on multiple overlaps define the twist λ_{ijkl} of the gerbe,
- the point part of hol_i defines the gerbe 1-form A_i
- the arrow part of hol_i defines the gerbe 2-form B_i,
- the natural transformation in the transition law for hol_i encodes the 1-form a_{ij} of the gerbe,
- the curvature of hol_i defines the gerbe curvature 3-form H_i

and:

- the point part of the transition law for g_{ij} gives the gerbe transition law of the φ_{ij},
- the arrow part of the transition law for the g_{ij} gives the gerbe transition law of d_{ij},
- the coherence law for the natural transformation defining the transition law for the g_{ij} gives the gerbe transition law for the f_{ijk},
- the point part of the transition law for hol_i gives the gerbe transition law for A_i,
- the arrow part of the transition law for hol_i gives the gerbe transition law for B_i,
- the coherence law for the transition law of hol_i gives the gerbe transition law for a_{ij},
- the transition law for hol_i also gives the gerbe transition law for H_i.

This is our main result. In summary, we find that categorifying the notion of a principal bundle with connection and with strict structure 2-group gives a structure that includes as a special case that of a twisted nonabelian bundle gerbe with connection and curving, together with a notion of nonabelian surface holonomy when the fake curvature vanishes.

1.2 Nonabelian 2-Holonomies and Physics

The motivation for the work in this paper comes to a good part from certain open questions in theoretical physics, which we briefly indicate here.

In the context of 'M-theory' (the, only partially understood, expected 11-dimensional completion of string theory, which again is a widely studied candidate theory of quantum gravity and unification of forces) spatially 2-dimensional objects called membranes, or M2-branes, are fundamental, as are their magnetic duals, the M5-branes.

The general configuration of M2-branes and M5-branes involves membranes whose boundaries are attached to the 5-branes, generalizing the way how open strings may end on various types of branes. (In fact, in type II string theory, networks of (p, q)-strings may end on webs of (p, q)-5-branes and these configurations lift in M-theory to configurations of M2s on M5s.)

While the bulk of the membrane couples to the supergravity 3-form potential, its boundary, when ending on a 5-brane, couples to the 2-form that is part of the field content on that brane.

By comparison with the analogous situation for strings, a stack of coinciding 5-branes should carry a *nonabelian* 2-form. Therefore the boundary part of the action for a membrane ending on that stack should involve some notion of *nonabelian surface holonomy* with respect to the worldsheet of the membrane's boundary.

But a general definition and theory of nonabelian surface holonomy is essentially missing. It is well understood how *abelian* gerbes give rise to a notion of *abelian* surface holonomy. Recently abelian gerbes have been generalized to *nonabelian* gerbes, but for the latter so far no concept of surface holonomy is known. On the other hand, using 2-groups it is straightforward to come up with a local notion of surface holonomy. However, in order to construct a well defined action for a membrane ending on a stack of 5-branes, it is crucial to take care of global issues.

For these reasons the construction of a globally defined notion of nonabelian surface holonomy, which is the subject of this paper, is a necessary prerequisite for fully understanding the fundamental objects of M-theory.

This is interesting for ordinary field theory, quite independently of whether strings really exist or not: Configurations of membranes ending on 5-branes can alternatively be described in terms of certain (superconformal) field theories of (possibly nonabelian) 2-form fields defined on the six-dimensional worldvolume of 5-branes. When these field theories are compactified on a torus they give rise to (super)Yang-Mills theory in four dimensions. Interestingly, the famous Montonen-Olive $\mathrm{SL}(2, \mathbb{Z})$ duality of such 4-dimensional gauge theory should arise this way simply as the modular transformations on the internal torus which act as symmetries on the conformally invariant six-dimensional theory.

From this point of view nonabelian 2-form gauge theory in six dimensions appears as a tool for understanding (strongly coupled) ordinary gauge theory in four dimensions.

It is interesting to note that these six-dimensional theories require the curvature 3-form of the 2-form field to be Hodge-*self-dual*. In the nonabelian case this is subtle because this 3-form should obey the local transition laws of nonabelian gerbes, which are not (at least

not in any obvious way) compatible with Hodge-self-duality in general, since they involve corrections to a covariant transformation of the 3-form. The only *obvious* solution to this compatibility problem is to require the so-called '*fake curvature*' of the gerbe to vanish (but it is not known if this is the most general solution). If this is the case the 3-form field strength transforms covariantly and can hence consistenly be chosen to be self-dual.

A major theme of the present paper is to show that what we call 2-bundles with 2-connections and with strict structure 2-group gives rise to general nonabelian gerbes, but necessarily subject to the constraint of vanishing 'fake curvature'. This vanishing condition is a direct consequence of the very nature of strict 2-groups and tightly related to our ability to construct a global nonabelian surface-holonomy.

For these reasons 2-bundles with 2-connections can be expected to play a role for the description of nonabelian gauge theories in four and in six dimensions.

An overview of the relation between four- and six-dimensional conformal field theories is given in [17]. A list of references on the physics of 5-branes is given in [9]. For (p, q) webs of 5-branes see [18, 19, 20].

Structure of this paper. We approach the subject in §2.1 (p.9) by informally considering some general aspects of gauge theory, holonomy, and its higher dimensional generalization using categorification.

This leads us to recall some aspects of 2-bundles in §2.2 (p.21) and to continue their study by working out the local meaning of the existence of 2-transitions.

After 2-bundles without 2-connection have thus been understood, the problem of globally constructing 2-holonomy in a 2-bundle is tackled in §2.3 (p.33) by a thorough investigation of local connections and holonomies on path spaces, which allows us to construct the local 2-functor $\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$ in terms of a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B on M.

With this functor in hand it is straightforward to categorify the notion of connection in a bundle and obtain the definition of a (global!) 2-connection in §2.4 (p.57), where we again translate the abstract diagrams into local data and find that the 2-transitions for the 2-connection define, under certain conditions, the cocycle data of a nonabelian gerbe with connection and 'curving'.

A summary and discussion of these constructions and results is given in §3 (p.64).

2. 2-Connections

2.1 Higher Gauge Theory

Before entering the details of the theory of 2-bundles with 2-connections, it is helpful to recall some general facts about the nature of gauge theory and its generalization to the parallel transport of higher dimensional objects. (See also [4].)

2.1.1 Algebra from Topology

Ordinary gauge theory describes how 0-dimensional particles transform as we move them along 1-dimensional paths. It is natural to assign Lie group elements to each path:

$$\bullet \xrightarrow{g} \bullet$$

The reason is that composition of paths then corresponds to multiplication in the group:

$$\bullet \xrightarrow{g} \bullet \xrightarrow{g'} \bullet$$

while reversing the direction of a path corresponds to taking inverses:

$$\bullet \xleftarrow{g^{-1}} \bullet$$

and the associative law makes the holonomy along a triple composite unambiguous:

$$\bullet \xrightarrow{g} \bullet \xrightarrow{g'} \bullet \xrightarrow{g''} \bullet$$

In short, the topology dictates the algebra.

To *really* let the topology dictate the algebra, we should replace the Lie group by a 'smooth groupoid': a groupoid in some convenient category of smooth spaces. Mackaay and Picken [15] have noted that for any manifold M there is a smooth groupoid $\mathcal{P}_1(M)$, the **path groupoid**, for which:

- objects are points $x \in M$
- morphisms are thin homotopy classes of smooth paths $\gamma : [0, 1] \to M$ such that $\gamma(t)$ is constant near $t = 0, 1$.

For any Lie group G, a principal G-bundle $P \to M$ gives a smooth groupoid $\mathrm{Trans}(P)$, the **transport groupoid**, for which:

- objects are torsors P_x for $x \in M$
- morphisms are torsor morphisms.

Via parallel transport, any connection on P gives a smooth functor called its **holonomy**

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to \mathrm{Trans}(P)\ . \tag{2.1}$$

A trivialization of P makes $\mathrm{Trans}(P)$ equivalent to G, so it gives:

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to G\,.$$

Now suppose we wish to do something similar for 'strings' that move along surfaces. Naively we might wish our holonomy to assign a group element to each surface like this:

$$\begin{tikzcd} \bullet \arrow[r, bend left=40, ""{name=U, below}] \arrow[r, bend right=40, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "g"] \end{tikzcd}$$

There are two obvious ways to compose surfaces of this sort, vertically:

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=M2, below}] \arrow[r, bend right=60, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=M, "g"] \arrow[Rightarrow, from=M2, to=D, "g'"] \end{tikzcd}$$

and horizontally:

$$\begin{tikzcd} \bullet \arrow[r, bend left=40, ""{name=U, below}] \arrow[r, bend right=40, ""{name=D}] & \bullet \arrow[r, bend left=40, ""{name=U2, below}] \arrow[r, bend right=40, ""{name=D2}] & \bullet \arrow[Rightarrow, from=U, to=D, "g"] \arrow[Rightarrow, from=U2, to=D2, "g'"] \end{tikzcd}$$

Suppose that both of these correspond to multiplication in the group G. Then to obtain well-defined holonomy for this surface regardless of whether we do vertical or horizontal composition first:

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=M2, below}] \arrow[r, bend right=60, ""{name=D}] & \bullet \arrow[r, bend left=60, ""{name=V, below}] \arrow[r, ""{name=N, above}, ""{name=N2, below}] \arrow[r, bend right=60, ""{name=E}] & \bullet \arrow[Rightarrow, from=U, to=M, "g_1"] \arrow[Rightarrow, from=M2, to=D, "g'_1"] \arrow[Rightarrow, from=V, to=N, "g_2"] \arrow[Rightarrow, from=N2, to=E, "g'_2"] \end{tikzcd}$$

we must have

$$(g_1 g_2)(g'_1 g'_2) = (g_1 g'_1)(g_2 g'_2).$$

This forces G to be abelian!

In fact, this argument goes back to a classic paper by Eckmann and Hilton [21]. Moreover, they showed that even if we allow G to be equipped with two products, say $g \circ g'$ for vertical composition and $g \cdot g'$ for horizontal, so long as both products share the same unit and satisfy this '**interchange law**':

$$(g_1 \circ g'_1) \cdot (g_2 \circ g'_2) = (g_1 \cdot g_2) \circ (g'_1 \cdot g'_2)$$

then in fact they must agree — so by the previous argument, both are abelian. The proof is very easy:

$$g \cdot g' = (g \circ 1) \cdot (1 \circ g') = (g \cdot 1) \circ (1 \cdot g') = g \circ g'.$$

Pursuing this approach, we ultimately get the theory of connections on 'abelian gerbes'. If $G = \mathrm{U}(1)$, such a connection looks locally like a 2-form - and it shows up naturally in string theory, satisfying equations very much like those of electromagnetism.

To go beyond this and get *nonabelian* higher gauge fields, we should let the topology dictate the algebra, and consider a connection that gives holonomies *both for paths and for surfaces.*

We can replace the path groupoid by the **path 2-groupoid**, where

- objects are points in M:

$$\bullet$$

- morphisms are smooth paths $\gamma\colon [0,1] \to M$ with $\gamma(\sigma)$ constant near $\sigma = 0, 1$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "\gamma"] & \bullet \end{tikzcd}$$

- 2-morphisms are thin homotopy classes of smooth maps $f\colon [0,1]^2 \to M$ with $f(\sigma, \tau)$ independent of τ near $\tau = 0, 1$ and constant near $\sigma = 0, 1$

$$\begin{tikzcd} \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "f"] \end{tikzcd}$$

Assume that for each path we have a holonomy taking values in some group G: where composition of paths corresponds to multiplication in G. Assume also that for each 1-parameter family of paths with fixed endpoints we have a holonomy taking values in some other group H: where vertical composition corresponds to multiplication in H:

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=M2, below}] \arrow[r, bend right=60, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=M, "h"] \arrow[Rightarrow, from=M2, to=D, "h'"] \end{tikzcd}$$

Next, assume that we can parallel transport an element $g \in G$ along a 1-parameter family of paths to get a new element $g' \in G$:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g", ""{name=U, below}] \arrow[r, bend right, "g'"', ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "h"] \end{tikzcd}$$

Now, the picture above suggests that we should think of h as a kind of 'arrow' or 'morphism' going from g to g'. We can use category theory to formalize this. However, in category theory, when a morphism goes from an object x to an object y, we think of the morphism as determining both its source x and its target y. The group element h does not determine g or g'. However, the pair (h, g) does. Thus it is useful to create a category

$\mathcal{G}$ where the set $\mathcal{G}^1$ of objects is just G, while the set $\mathcal{G}^2$ of morphisms consists of pairs $f = (h, g) \in H \times G$. Switching our notation to reflect this, we rewrite the above picture as

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow f"] & \bullet
\end{tikzcd}$$

and write $f\colon g \to g'$ for short. We have source and target maps

$$s, t\colon \mathcal{G}^2 \to \mathcal{G}^1$$

with $s(f) = g$ and $t(f) = g'$.

In this new notation we can vertically compose $f\colon g \to g'$ and $f'\colon g' \to g''$ to get $f \circ f'\colon g \to g''$, as follows:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, bend left=30, "\Downarrow f"] \arrow[r, phantom, bend right=30, "\Downarrow f'"] & \bullet
\end{tikzcd}$$

This is just composition of morphisms in the category $\mathcal{G}$. However, we can also horizontally compose $f_1\colon g_1 \to g_1'$ and $f_2\colon g_2 \to g_2'$ to get $f_1 \cdot f_2\colon g_1 g_2 \to g_1' g_2'$, as follows:

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_1'"'] \arrow[r, phantom, "f_1 \Downarrow"] & \bullet \arrow[r, bend left, "g_2"] \arrow[r, bend right, "g_2'"'] \arrow[r, phantom, "f_2 \Downarrow"] & \bullet
\end{tikzcd}$$

We assume this operation makes $\mathcal{G}^2$ into a group with the pair $(1, 1) \in H \times G$ as its multiplicative unit.

The good news is that now we can assume an interchange law saying this holonomy is well-defined:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, "g_1"] \arrow[r, "g_2" description] \arrow[r, bend right=60, "g_3"'] \arrow[r, phantom, bend left=30, "\Downarrow f_1"] \arrow[r, phantom, bend right=30, "\Downarrow f_1'"] & \bullet \arrow[r, bend left=60, "g_1'"] \arrow[r, "g_2'" description] \arrow[r, bend right=60, "g_3'"'] \arrow[r, phantom, bend left=30, "\Downarrow f_2"] \arrow[r, phantom, bend right=30, "\Downarrow f_2'"] & \bullet
\end{tikzcd}$$

namely:

$$(f_1 \circ f_1') \cdot (f_2 \circ f_2') = (f_1 \cdot f_2) \circ (f_1' \cdot f_2') \tag{2.2}$$

without forcing either $\mathcal{G}^2$ or $\mathcal{G}^1$ to be abelian! Instead, $\mathcal{G}^2$ is forced to be a semidirect product of the groups G and H.

The structure we are rather roughly describing here is in fact already known to mathematicians under the name of a 'categorical group' [22, 23]. The reason is that $\mathcal{G}$ turns out to be a category whose set of objects $\mathcal{G}^1$ is a group, whose set of morphisms $\mathcal{G}^2$ is a group, and where all the usual category operations are group homomorphisms. To keep the

terminology succinct and to hint at generalizations to still higher-dimensional holonomies, we prefer to call this sort of structure a '2-group'. Moreover, we shall focus most of our attention on 'Lie 2-groups', where $\mathcal{G}^1$ and $\mathcal{G}^2$ are Lie groups and all the operations are smooth. More details on Lie 2-groups can be found in a paper by Baez and Lauda [2].

A Lie 2-group $\mathcal{G}$ amounts to the same thing as a 'Lie crossed module': a pair of Lie groups G and H together with a homomorphism $t\colon H \to G$ and an action α of G on H satisfying a couple of compatibility conditions. The idea is to let $G = \mathcal{G}^1$, let H be the subgroup of $\mathcal{G}^2$ consisting of morphisms with source equal to 1, and let t be the map sending each such morphism to its target. The action α is defined by letting $\alpha(g)f$ be this horizontal composite:

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow 1_g"] & \bullet \arrow[r, bend left, "1"] \arrow[r, bend right, "t(f)"'] \arrow[r, phantom, "\Downarrow f"] & \bullet \arrow[r, bend left, "g^{-1}"] \arrow[r, bend right, "g^{-1}"'] \arrow[r, phantom, "\Downarrow 1_{g^{-1}}"] & \bullet
\end{tikzcd}$$

It appears that one can develop a full-fledged theory of bundles, connections, curvature, and so on with a Lie 2-group taking the place of a Lie group. So far most work has focused on the special case when G is trivial and $H = \mathrm{U}(1)$, using the language of $\mathrm{U}(1)$ gerbes [24, 25, 26, 27, 28, 15]. Here, however, we really want H to be nonabelian. Some important progress in this direction can be found in Breen and Messing's paper on the differential geometry of nonabelian gerbes [11]. While they use different terminology, their work basically develops the theory of connections and curvature for Lie 2-groups where H is an arbitrary Lie group, $G = \mathrm{Aut}(H)$ is its group of automorphisms, t sends each element of H to the corresponding inner automorphism, and the action of G on H is the obvious one. We call this sort of Lie 2-group the 'automorphism 2-group' of H. Luckily, it is easy to extrapolate the whole theory from this case.

In particular, for any Lie 2-group $\mathcal{G}$ one can define the notion of a 'principal 2-bundle' having $\mathcal{G}$ as its gauge 2-group; this has recently been done by Bartels [5]. One can also define connections and curvature for these principal 2-bundles.

One of our goals in this paper is to show that given a connection of this sort (satisfying certain conditions), we may define holonomies for paths and surfaces which behave just as one would like. The other is to relate this notion of 2-connection to that of a connection on the space $\mathcal{P}_1(M)$ whose points are paths in M. A connection on $\mathcal{P}_1(M)$ assigns a holonomy to any path in $\mathcal{P}_1(M)$, but such a path traces out a surface in M. Such a connection thus gives a concept of (nonabelian) surface holonomy, which however will depend on the *parameterization* of the surface unless we impose extra conditions.

Intuitively it is clear that these two points of view should be closely related, but little is known about the details of this relation. Motivated by the recent discovery [9] that a certain consistency condition for surface holonomy appearing in the loop space approach is discussed also in the literature on 2-groups [7], while other such consistency conditions have exclusively been discussed in the loop space context [8], the aim here is to clarify this issue.

Hence in analogy to the situation (2.1) for ordinary bundles with ordinary connections, in the following we want to make precise the following statement:

A principal $\mathcal{G}$-2-bundle $P \to B$ gives a smooth 2-groupoid $\mathrm{Trans}(P)$ where

- objects are 2-torsors P_x
- morphisms are 2-torsor morphisms $f\colon P_x \to P_y$
- 2-morphisms are 2-torsor 2-morphisms $\theta\colon f \Rightarrow g$.

Via parallel transport, a 2-connection on P gives a smooth 2-functor called its **holonomy**:

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathrm{Trans}(P)\ .$$

A trivialization of P makes $\mathrm{Trans}(P)$ equivalent to $\mathcal{G}$, so that it gives

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}\,.$$

In order to understand how such a structure is included in the framework of categorified bundles, it is very helpful to use a technique called 'internalization'.

2.1.2 Internalization

The crucial concept for categorification is 'internalization'. Ehresmann and Lawvere showed [29, 30] how to 'internalize' concepts by defining them in terms of commutative diagrams (see section 2 of [3] for more details):

A **small category**, say C, has a <u>set</u> of objects $\mathrm{Ob}(C) \equiv C^1$, a <u>set</u> of morphisms $\mathrm{Mor}(C) \equiv C^2$, source and target <u>functions</u>

$$s, t\colon \mathrm{Ob}(C) \to \mathrm{Mor}(C),$$

a composition <u>function</u>

$$\circ\colon \mathrm{Mor}(C)_s{\times}_t\mathrm{Mor}(C) \to \mathrm{Mor}(C)$$

and an identity–assigning <u>function</u>

$$\mathrm{id}\colon \mathrm{Ob}(C) \to \mathrm{Mor}(C)$$

making diagrams commute which describe associativity of composition and behaviour of source and target maps under composition.

Internalization means letting these diagrams live within some category K:

A **category in** K, say C, has an <u>object</u> $\mathrm{Ob}(C) \in K$, an <u>object</u> $\mathrm{Mor}(C) \in K$, source and target <u>morphisms</u>

$$s, t\colon \mathrm{Ob}(C) \to \mathrm{Mor}(C),$$

a composition <u>morphism</u>

$$\circ\colon \mathrm{Mor}(C)_s{\times}_t\mathrm{Mor}(C) \to \mathrm{Mor}(C)$$

and an identity-assigning <u>morphism</u>

$$\mathrm{id}\colon \mathrm{Ob}(C) \to \mathrm{Mor}(C)$$

making the above diagrams commute in K.

Similarly we can define **functors in** K and **natural transformations in** K, obtaining a 2-category $\boldsymbol{K}\mathbf{Cat}$. We can also define **groups in** K and **homomorphisms in** K, obtaining a category $\boldsymbol{K}\mathbf{Grp}$.

2.1.3 Smooth Categories, 2-Groups, and Lie 2-Groups

Using the above, we can categorify concepts from differential geometry with the help of internalization:

- A **smooth category** (called **2-space** in the following) is a category in Diff, the category of smooth spaces with smooth maps between them.
- A **strict 2-group** (or **categorical group**) is a category in Grp, the category of groups with homomorphisms between them.
- A **strict Lie 2-group** is a category in LieGrp, the category of Lie groups.

Important examples of strict 2-groups are the following:

1) Any abelian group A gives a strict 2-group with one object and A as the automorphisms of this object. Lie 2-groups of this kind will be structure 2-groups of 2-bundles giving rise to *abelian gerbes.*

2) Any category C gives a 2-group $\mathrm{Aut}(C)$ whose objects are equivalences $f: C \to C$ and whose morphisms are natural isomorphisms between these.

3) A group H is a category with one object and all morphisms invertible. In this case, 2) gives a strict 2-group $\mathrm{Aut}(H)$ whose objects are automorphisms of H and whose morphisms from f to f' are elements $k \in H$ with $f'(h) = kf(h)k^{-1}$.

4) Any Lie group H gives a strict Lie 2-group $\mathrm{Aut}(H)$ defined as in 3) but with everything smooth. Lie 2-groups of this sort will be structure 2-groups of 2-bundles that give rise to *nonabelian gerbes.*

Properties of Strict 2-Groups. When it comes to explicit descriptions the most important fact about strict 2-groups is that they are characterized by *crossed modules*, in a way to be made precise below in prop. 2.1.

Definition 2.1 *A* **Lie crossed module** *is a quadruple* (G, H, t, α) *consisting of Lie groups* G *and* H*, a homomorphism* $t : H \to G$*, and an action of* G *on* H *(that is, a homomorphism* $\alpha : G \to \mathrm{Aut}(H)$*) satisfying*

$$t(\alpha(g)(h)) = g\,t(h)\,g^{-1}$$

and

$$\alpha(t(h))\big(h'\big) = h\,h'\,h^{-1}$$

for all $g \in G$ *and* $h, h' \in H$*.*

(*cf.* [4], definition 3.)

Definition 2.2 *A* **differential crossed module** *is a quadruple* $(\mathfrak{g}, \mathfrak{h}, dt, d\alpha)$ *consisting of Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$*, a homomorphism* $dt : \mathfrak{h} \to \mathfrak{g}$*, and an action of* $\mathfrak{g}$ *on* $\mathfrak{h}$ *(that is, a homomorphism* $d\alpha : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$*) satisfying*

$$dt(d\alpha(x)(y)) = [x, dt(y)]$$

and

$$d\alpha(dt(y))\big(y'\big) = \big[y, y'\big] \tag{2.3}$$

for all $x \in \mathfrak{g}$ *and* $y, y' \in \mathfrak{h}$*.*

For convenience we will also write

$$d\alpha(x)\big(x'\big) \equiv \big[x, x'\big]$$

for $x, x' in \mathfrak{g}$*.*

(*cf.* [4], definition 15.)

Now the relation between crossed modules and strict 2-groups is the following:

Proposition 2.1 *Every strict 2-group comes from a crossed module (def. 2.1)* (G, H, α, t) *such that 2-group elements are labeled by pairs*

$$(g, h) \qquad \text{with } g \in G \text{ and } h \in H\,,$$

such that the source d_0 *and target* d_1 *are given by*

$$\begin{aligned} d_0((g,h)) &= g \\ d_1((g,h)) &= t(h)\,g \end{aligned} \tag{2.4}$$

and such that horizontal and vertical composition is given by

$$\begin{aligned} (g,h)\cdot(g',h') &= (gg', h\,\alpha(g)\big(h'\big)) \\ (g,h)\circ(t(h)\,g, h^1) &= (g, h'h)\,, \end{aligned} \tag{2.5}$$

respectively.

Proof. The proof is given for instance in [22]. □

It is the property (2.4) of strict 2-groups that leads to the constraint of vanishing fake curvature in 2-bundles with 2-connections, as has first been explicitly disussed in [7] using lattice 2-gauge theory, and as we will derive using path space connections in §2.3.4 (p.48).

We end this section by mentioning a couple of simple facts about 2-groups that are needed in calculations throughout this paper:

Definition 2.3 *There are two kinds of* **inverses** *for 2-groups, those with respect to the horizontal and those with respect to the vertical product. The inverse with respect to the horizontal product will be denoted by* $(\cdot)^{-1}$*, while that with respect to the vertical product by* $(\cdot)^r$ *(for "*r*eversion").*

Corollary 2.1 *The following lists a couple of useful facts about strict 2-groups:*

1. *With respect to the horizontal product (2.5) of a strict 2-group the* **identity** *element is*

$$1 = (1,1)$$

and the **horizontal inverse** *(def. 2.3) of* (g,h) *is*

$$(g,h)^{-1} = (g^{-1}, \alpha(g^{-1})(h^{-1}))\,.$$

The **vertical identity** *with base* g *is*

$$1_g \equiv (g,1) \tag{2.6}$$

and the **vertical inverse** *(def. 2.3) of* (g,h) *is*

$$(g,h)^r = \left(t(h)\,g, h^{-1}\right)\,.$$

2. **Horizontal conjugation** *with a vertical identity (2.6) yields*

$$1_g \cdot (g',h') \cdot (1_g)^{-1} = \left(g\,g'\,g^{-1}, \alpha(g)(h')\right)$$

3. *The 2-group elements with source and target the identity in* $1 \in G$ *form an abelian subgroup. By prop. 2.1 this are elements of the form* $(1,a)$ *with* $t(a) = 1$*. Equivalently* $\ker(t) \in H$ *is a normal abelian subgroup of* H*.*

Proof.

3.: This is a special case of a famous argument by Eckmann-Hilton (*cf.* [2], p. 55).

□

We conclude our review of categorified gauge theory by briefly discussion nonabelian gerbes in the next subsection.

2.1.4 Nonabelian Gerbes

Categorification, while being a systematic procedure, can usually be performed in several ways, in particular when there are several equivalent formulations of the original structure to be categorified.

The 2-bundles to be discussed in the present paper arise as the categorification of ordinary bundles, being maps $E \xrightarrow{p} B$ from some total space to a base space. On the other hand, an ordinary bundle is alternatively characterized by its sheaf of sections. Gerbes,

which go back to work by Giraud, arise roughly as the categorification of the notion of a sheaf [24, 31, 11].

The theory of abelian gerbes has been motivated by the desire to find a higher dimensional version of the fact that line bundles over M provide a geometric realization of elements of $H^2(M,\mathbb{Z})$. Abelian gerbes provide a geometric realization of the elements of $H^3(M,\mathbb{Z})$.

As a more concrete description of gerbes than the original one in the language of algebraic geometry, Murray introduced (abelian) 'bundle gerbes' [14], Chatterjee introduced (abelian) 'gerbs' [32] and recently Aschieri, Cantini and Jurčo studied the nonabelian generalization of bundle gerbes [12].

'Bundle gerbes' involve bundles over bundles of paths over a base space and hence can be handled in terms of differential geometry, which is more conveniently applied to physics (e.g. [25, 13, 33]) than algebraic geometry. Their relation to path spaces already indicates that (bundle) gerbes should be closely related to 2-bundles over 2-spaces, since the archetypical 2-space is a path space. In the following we shall study how close this connection really is.

The term 'gerb' introduced by Chatterjee is supposed to refer to the description of gerbes in terms of local transition functions and Čhech cocycles. This is the most elementary description and also the only perspective on gerbes considered in the present paper, a good part which is devoted to working out the description of 2-bundles with 2-connections in terms of local data and showing that the 2-transition laws reproduce the cocycle descrition of nonabelian gerbes. In order to be self-contained, this cocycle data is listed here.

For nonabelian gerbes this was derived in terms of bundle gerbes in [12]. Our notation follows that in [13]. For vanishing 'twist' the same in different notation was first given in [11].

The cocycle data of a nonabelian gerbe consists of

- a base space M
- a good cover U of M (with $U^{[n]}$ denoting the space of n-fold intersections of patches in U)
- a crossed module (G, H, α, t) with differential crossed module $(\mathfrak{g}, \mathfrak{h}, d\alpha, dt)$ (usually considered to be an automorphism crossed module)
- transition functions

$$\begin{aligned} U^{[2]} &\to \Omega^0(M,G) \\ (x,i,j) &\mapsto g_{ij}(x) \end{aligned} \tag{2.7}$$

- connection 1-forms

$$\begin{aligned} U^{[1]} &\to \Omega^1(M,\mathfrak{g}) \\ (x,i) &\mapsto A_i(x) \end{aligned} \tag{2.8}$$

- curving 2-forms

$$\begin{aligned} U^{[1]} &\to \Omega^2(M,\mathfrak{h}) \\ (x,i) &\mapsto B_i(x) \end{aligned} \tag{2.9}$$

- transition transformation 0-forms

$$\begin{aligned} U^{[3]} &\to \Omega^0(M,H) \\ (x,i,j,k) &\mapsto f_{ijk}(x) \end{aligned} \tag{2.10}$$

- connection transformation 1-forms

$$\begin{aligned} U^{[2]} &\to \Omega^1(M,\mathfrak{h}) \\ (x,i,j) &\mapsto a_{ij}(x) \end{aligned} \tag{2.11}$$

- curving transformation 2-forms

$$\begin{aligned} U^{[2]} &\to \Omega^2(M,\mathfrak{h}) \\ (x,i,j) &\mapsto d_{ij}(x) \end{aligned} \tag{2.12}$$

- twist p-forms

$$\begin{aligned} U^{[4]} &\to \Omega^0(M,\ker(t)\subset H) \\ (x,i,j,k,l) &\mapsto \lambda_{ijkl}(x) \\[1em] U^{[3]} &\to \Omega^1(M,\ker(dt)\subset \mathfrak{h}) \\ (x,i,j,k) &\mapsto \alpha_{ijk}(x) \\[1em] U^{[2]} &\to \Omega^2(M,\ker(dt)\subset \mathfrak{h}) \\ (x,i,j) &\mapsto \beta_{ij}(x) \\[1em] U^{[1]} &\to \Omega^3(M,\ker(dt)\subset \mathfrak{h}) \\ (x,i) &\mapsto \gamma_i(x) \end{aligned} \tag{2.13}$$

such that the following transition laws are satisfied:

- transition law for the transition functions

$$\phi_{ij}(x)\,\phi_{jk}(x) = t(f_{ijk}(x))\,\phi_{ik}(x)\;,\;\;\forall (x,i,j,k)\in U^{[3]} \tag{2.14}$$

- transition law for the connection 1-forms

$$A_i(x) + dt\big(a_{ij(x)}\big) \;=\; \phi_{ij}(x)\,A_j(x)\,\phi_{ij}^{-1}(x) + \phi_{ij}(x)\,(\mathbf{d}\phi_{ij}^{-1})(x)\;,\;\;\forall (x,i,j)\in U^{[2]} \tag{2.15}$$

- transition law for the curving 2-forms

$$B_i(x) = \alpha(\phi_{ij}(x))(B_j(x)) + k_{ij}(x) - d_{ij}(x) - \beta_{ij}(x) \ , \quad \forall (x,i,j) \in U^{[2]} \, . \tag{2.16}$$

- transition law for the curving transformation 2-forms

$$d_{ij} + \phi_{ij}(d_{jk}) = f_{ijk}\, d_{ik}\, f_{ijk}^{-1} + f_{ijk} d\alpha(dt(B_i) + F_{A_i})\, f_{ijk}^{-1} \, , \quad \forall (x,i,j) \in U^{[2]} \tag{2.17}$$

In addition to these there are what for reasons explained in §2.2 (p.21) and §2.4 (p.57) we here shall call *coherence laws*, since they ensure that compositions of the above transformations are well defined:

- coherence law for the transformators of the transition functions

$$f_{ikl}^{-1}(x)\, f_{ijk}^{-1}(x)\ \alpha(\phi_{ij}(x))(f_{jkl}(x))\, f_{ijl}(x) = \lambda_{ijkl}(x) \ , \quad \forall (x,i,j,k,l) \in U^{[4]} \tag{2.18}$$

- coherence law for the transformators of the connection 1-form

$$\alpha_{ijk} = a_{ij} + \phi_{ij}(a_{jk}) - f_{ijk}\, a_{ik}\, f_{ijk}^{-1} - f_{ijk} d f_{ijk}^{-1} - f_{ijk} d\alpha(A_i)\left(f_{ijk}^{-1}\right) \, . \tag{2.19}$$

Finally the *curvature 3-form* of the nonabelian gerbe is defined as

$$H_i \equiv d_{A_i} B_i + \gamma_i \tag{2.20}$$

and its transformation law is

$$H_i = \phi_{ij}(H_j) - \mathbf{d} d_{ij} - [a_{ij}, d_{ij}] - d\alpha(dt(B_i) + F_{A_i})(a_{ij}) - d\alpha(A_i)(d_{ij}) \, . \tag{2.21}$$

2.2 2-Bundles (without Connection)

2-bundles have been defined by Toby Bartels [5] by categorification of the concept of an ordinary bundle.

An orinary bundle consists of two sets, the **total space** E and the **base space** B, together with a **projection map**

$$E \xrightarrow{p} B .$$

We can think of p as a morphism in the category Set, whose objects are sets and whose morphisms are functions between sets.

Usually one wants E and B not to be general sets, but to be sets that are smooth spaces. There is a subcategory Diff of Set whose objects are smooth spaces and whose morphisms are smooth maps between these. A *bundle in* Diff is hence a smooth space E and a smooth space B together with a smooth map $E \xrightarrow{p} B$.

The category Set as well as its sub-category Diff are 1-categories. This means that they have objects and (1-)morphisms going between objects, but no nontrivial 2-morphisms going between 1-morphisms. We get a categorified notion of bundle by considering a *bundle in* C, for C a proper 2-category which has objects, 1-morphisms between objects and 2-morphisms between 1-morphisms (but no nontrivial 3-morphisms between 2-morphisms).

The natural choice for a 2-category replacement for the 1-category Set is the 2-category Cat, the 2-category whose objects are categories, whose morphisms are functors and whose 2-morphisms are natural transformations.

Hence a *bundle in* Cat is a category E, a category B and a functor $E \xrightarrow{p} B$.

As with ordinary bundles, usually one wants categorified bundles to have some smoothness property. In analogy to the subcategory Diff of Set there should be a sub-2-category $\mathfrak{C}$ of Cat whose objects are 'smooth categories' in an appropriate sense. The notion of smooth category, called *2-space*, is itself obtained by internalization:

Definition 2.4 *A* **2-space** *is a category internalized in* Diff_∞*, where* Diff_∞ *is the category of (possibly infinite dimensional) smooth spaces with smooth maps between them as morphisms. A* **2-map** *is a smooth functor between two 2-spaces.*

We write a 2-space S as $S = (S^1, S^2)$ with $S^1 = \mathrm{Ob}(S)$ the space of objects, also called the **point space** and $S^2 = \mathrm{Mor}(S)$ the space of morphisms, also called the **arrow space**. Being a category implies that there are smooth maps $d_0 : S^2 \to S^1$ and $d_1 : S^2 \to S^1$ which map each arrow in S^2 to its **source** and **target** point in S^1, respectively. Moreover, there is a smooth map which sends two composable arrows in S^2 to their composition in S^2.

Two important subsets of 2-spaces that play a role in this paper are 'trivial' and 'simple' 2-spaces:

Definition 2.5

- *A 2-space for which all morphisms are identity morphisms shall be called a* **trivial 2-space***.*

- *A 2-space for which the source and the target map coincide shall be called a* **simple 2-space**.

Smooth maps $f = (f^1, f^2)$ between 2-spaces, consist of an ordinary smooth map f^1 taking the point space smoothly to another point space, together with a smooth map f^2 taking arrows to arrows in such a way that it respects the action of the point map on the source and target of these arrows.

The strict 2-category $\mathfrak{C}$ that we are interested in is the 2-category of 2-spaces, whose objects are 2-spaces, whose morphisms are 2-maps and whose 2-morphisms are natural transformations between 2-maps.

This allows us to finally state the definition of a 2-bundle:

Definition 2.6 *A* **2-bundle** *is a* bundle in $\mathfrak{C}$, *i.e. the collection of*

- *a 2-space P (the* **total space***)*
- *a 2-space B (the* **base space***)*
- *a 2-map $p: P \to B$ (the* **projection***) .*

For applications in gauge theory, a bundle should be *locally trivializable.* This is the content of the next subsection.

2.2.1 Locally trivializable 2-Bundles

We shall be interested in *locally trivializable* 2-bundles. These require the notion of a 2-cover. Ordinarily, a locally trivializable bundle is a bundle $E \xrightarrow{p} B$ together with a cover $U = \bigsqcup_{i\in I} U_i$ being the disjoint union of open contractible subsets $U_i \subset B$ with $\bigcup\limits_{i\in I} U_i = B$, such that the restriction to any of the U_i yields a bundle isomorphic to a trivial bundle. In the context of 2-bundles this is implemented as follows:

2-covers.

Definition 2.7 *Given 2-spaces S and B, the 2-space S is called a* **sub-2-space** $S \subset B$ *of B if the point and arrow spaces of S are subsets of the point and arrow spaces of B and if the source, target and composition maps of S are the restriction of these maps on B to these subsets.*

Definition 2.8 *Given a 2-space B and a set of sub-2-spaces (def. 2.7) $\{U_i\}_{i\in I}$, $U_i \subset B$, their* **union of 2-spaces** *denoted by $\bigcup\limits_{i\in I} U_i \subset B$ is defined to be the sub-2-space of B (def. 2.7) whose point space is the union of the point spaces of the U_i and whose arrow space is the free space of morphisms generated under composition by the union of all morphisms in the arrow spaces of the U_i.*

Similarly the **intersection of 2-spaces** $\bigcap\limits_{i\in I} U_i \subset B$ *is defined to be the 2-space $\left(\bigcap\limits_{i\in I} U_i^1, \bigcap\limits_{i\in I} U_i^2\right)$ whose point space is the intersection of all the U_i^1 and whose arrow space is the intersection of all the U_i^2*

Note that the notion of union of sub-2-spaces depends on the total 2-space B, from which the union inherits its source, target and composition map.

Definition 2.9 *A* **2-cover** U *of a 2-space* B *is a* disjoint union *of sub-2-spaces* $\bigsqcup_{i\in I} U_i$ *(def. 2.7) such that the point and arrow space of each* U_i *is open and contractible and such that the union (def. 2.8) of all* U_i *is* B, $\bigcup_{i\in I} U_i = B$.

Every 2-cover is equippend with a 2-map

$$U \xrightarrow{j} B \tag{2.22}$$

that restricts on each U_i to the inverse of the restriction map of B to U_i.

Definition 2.10 *Given a manifold* M *the 2-space* $(M, \mathrm{L}M)$ *whose point space is* M *and whose arrows space is* $\mathrm{L}M$*, the space of free loops over* M *with the obvious source, target and composition maps, is called the* **free loop 2-space over** M.

Given any ordinary cover $\{U_i^1\}_{i\in I}$ *of* M *by 1-spaces* U_i^1*, the 2-space obtained as the union (def. 2.8) with respect to* $(M, \mathrm{L}M)$ *of all free loop 2-spaces over the* U_i^1 *is called a* **local free loop 2-space** *over* M *with respect to the cover* $\{U_i^1\}_{i\in I}$*:*

$$\bigcup_{i\in I} (U_i^1, U_i^2 \equiv \mathrm{L}U_i^1) \subset (M, \mathrm{L}M)\,.$$

Definition 2.11 *Given a 2-cover* U*, one will often need the spaces of* **multiple intersections** *of the* U_i*. We denote by* $U^{[n]}$ *the 2-space that is the disjoint union of the* n*-fold 2-space intersections (def. 2.8) of* U_i*:*

$$U^{[n]} = \bigsqcup_{i_1, i_2, \ldots, i_n \in I} U_{i_1} \cap U_{i_2} \cap \ldots \cap U_{i_n}\,.$$

Each 2-space of multiple intersections comes with 2-maps

$$\begin{array}{ccc} U^{[n]} & \overset{j_{01\cdots(k-1)(k+1)\cdots n}}{\longrightarrow} & U^{[n-1]} \\ (i_1, i_2, \ldots, i_n, x \xrightarrow{\gamma} y) & \mapsto & (i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n, x \xrightarrow{\gamma} y) \end{array} \tag{2.23}$$

that forget about the k-th member of the multiple intersection.

Locally trivializable 2-Bundles. With the notion 2-cover in hand, we can now state the definition of a locally trivializable 2-bundle:

Definition 2.12 *A* **locally trivializable 2-bundle** *is a 2-bundle (def. 2.6) together with a 2-space* F *called the* **local fiber** *such that there is a 2-cover (def. 2.9)* $\{U_i\}_{i\in I}$ *of the base 2-space* B *for which there exist smooth equivalences*

$$p^{-1}U_i \xrightarrow{t_i} U_i \times F$$

such that the diagram

$$
\begin{tikzcd}
p^{-1}U_i \arrow[rr, "t_i"] \arrow[dr, "p|_{p^{-1}U_i}"'] & & U_i \times F \arrow[dl] \\
& U_i &
\end{tikzcd}
$$

commutes for all $i \in I$ up to natural isomorphism.

Note that if $\{t_i\}_{i\in I}$ is a local trivialization, then so is $\{t'_i\}_{i\in I}$ with t'_i naturally isomorphic to t_i. Denote by $[t_i]$ the equivalence class of t_i under natural isomorphisms.

Definition 2.13 *The 2-cover $\{U_i\}_{i\in I}$ together with the set $\{[t_i]\}_{i\in I}$ is called a* **local trivialization**.

This definition is concise and elegant, but rather abstract. In §2.2.2 (p.25) we translate its meaning into transition laws for local data specifying the 2-bundle. In order to do so, first *transition functions* need to be extracted from a local trivialization:

By composing the local trivializations and their weak inverses on double intersections U_{ij} one gets autoequivalences of $U_{ij} \times F$ of the form

$$U_{ij} \times F \xrightarrow{\bar{t}_i \circ t_j} U_{ij} \times F$$

and similarly for other index combinations.

Definition 2.14 *A locally trivializable 2-bundle (def. 2.12) with a local trivialization (def. 2.13)*

- *where all autoequivalences $U_{ij} \times F \xrightarrow{\bar{t}_i \circ t_j} U_{ij} \times F$ act trivially on the U_{ij} factor, so that*

$$\bar{t}_i \circ t_j = \mathrm{id}_{U_{ij}} \times g_{ij} \,,$$

- *where F is a 2-group $\mathcal{G}$,*
- *and where the g_{ij} act by left horizontal 2-group multiplication on F*

we say that our 2-bundle is a **principal $\mathcal{G}$-2-bundle** *and that*

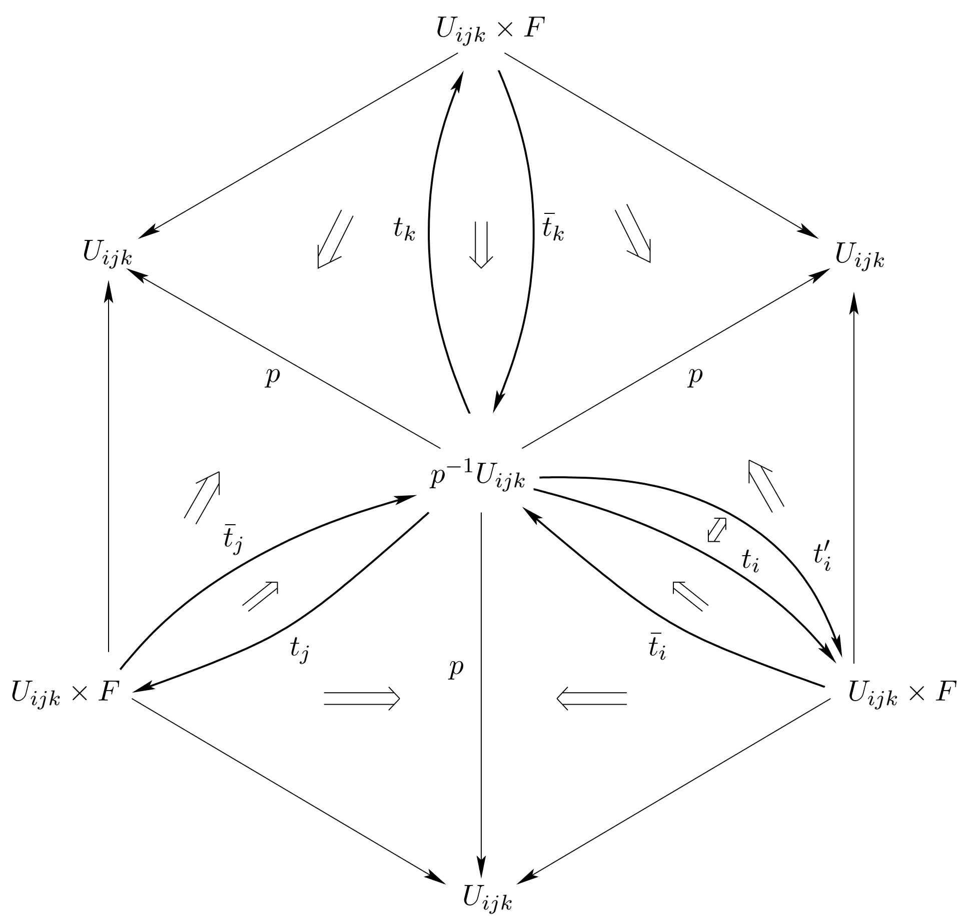
$$
\begin{aligned}
U^{[2]} &\xrightarrow{g} \mathcal{G} \\
U_{ij} &\mapsto g_{ij}
\end{aligned}
$$

is the **transition function**.

Note that according to def 2.13 each g_{ij} involves *choosing* maps t_i, t_j from the equivalence classes $[t_i], [t_j]$ of the local trivialization. This additional freedom gives rise to the modification of the transition law in a principal $\mathcal{G}$-2-bundle as compared to that of an ordinary principal bundle. This is discussed in the next section.

2.2.2 2-Transitions in Terms of local Data

Consider a triple intersection $U_{ijk} = U_i \cap U_j \cap U_k$ in a principal $\mathcal{G}$-2-bundle (def. 2.14). The existence of the local trivialization implies that the following diagram 2-commutes (all morphisms here are 2-maps and all 2-morphisms are natural isomorphisms between these):



Compared to the analogous diagram for an uncategorified bundle two important new aspects are that the barred morphisms are inverses-up-to-isomorphism of the local trivializations and that the local trivialization itself is unique only up to natural isomorphisms (def. 2.13). The latter is indicated by the presence of an arrow denoting a trivialization t'_i naturally isomorphic to t_i.

From the diagram it is clear that the usual transition law $g_{ij}g_{jk} = g_{ik}$ here becomes a natural isomorphism called a 2-transition, which was first considered in [5] for the special case of trivial base 2-spaces (def. 2.5), but which directly generalizes to arbitrary base 2-spaces:

Definition 2.15 *Given a base 2-space B with cover $U \xrightarrow{j} B$ a* **2-transition** *is*

- *a 2-map*

$$U^{[2]} \xrightarrow{g} \mathcal{G}$$

called the **transition function***,*

- *and a natural isomorphism* f

$$\begin{array}{c} U^{[3]} \xrightarrow{j_{02}} U^{[2]} \xrightarrow{g} \mathcal{G} \\ \overset{f}{\Longrightarrow} \\ U^{[3]} \xrightarrow{\overset{\vee}{U}^{[3]}} U^{[3]} \times U^{[3]} \xrightarrow{j_{01} \times j_{12}} U^{[2]} \times U^{[2]} \xrightarrow{g \times g} \mathcal{G} \times \mathcal{G} \xrightarrow{m} \mathcal{G}\,, \end{array}$$

(which expressed the categorification of the ordinary transition law $g_{ij}g_{jk} = g_{ik}$*), together with the coherence law for* f *enforcing the associativity of the product* $g_{ij}g_{jk}g_{kl}$*,*

- *and a natural isomorphism*

$$\begin{array}{c} U \xrightarrow{j_{00}} U^{[2]} \xrightarrow{g} \mathcal{G} \\ \overset{\eta}{\Longrightarrow} \\ U \xrightarrow{\hat{U}} 1 \xrightarrow{i} \mathcal{G}\,. \end{array}$$

(expressing the categorification of the ordinary $g_{ii} = 1$*) together with its coherence laws.*

In the above $U^{[3]} \xrightarrow{\overset{\vee}{U}^{[3]}} U^{[3]} \times U^{[3]}$ denotes the diagonal embedding of $U^{[3]}$ in its second tensor power and m denotes the horizontal multiplication (functor) in the 2-group $\mathcal{G}$. The maps $j_{\dots}$ have been defined in (2.23).

In terms of local functions this means the following:

Proposition 2.2 *A 2-transition (def. 2.15) on a* $\mathcal{G}$*-2-bundle with base 2-space being a simple 2-space (def. 2.5) and* $\mathcal{G}$ *a strict 2-group induces the transition law (2.14) of a nonabelian gerbe.*

Proof.

The existence of the natural isomorphism means that there is a map

$$\begin{aligned} (U^{[3]})^1 &\xrightarrow{f} \mathcal{G}^2 \\ (x,i,j,k) &\mapsto f_{ijk}(x)\ , \end{aligned}$$

with the property

$$g^2_{ik}(x) \circ f_{ijk}(x) = f_{ijk}(x) \circ (g^2_{ij}(x) \cdot g^2_{jk}(x))\,, \qquad \forall\,(x,i,j) \in U^{[2]}\,. \tag{2.24}$$

(Here $\circ$ denotes the vertical and $\cdot$ the horizontal product in the 2-group, see prop. 2.1)

For strict $\mathcal{G}$ the source/target matching condition implies that (again prop. 2.1)

$$t\big(f^2_{ijk}\big)\, g^1_{ik} = g^1_{ij} g^1_{jk}\,, \tag{2.25}$$

where we have decomposed the 2-group element

$$f_{ijk}(x) = (f^1_{ijk}(x)\,, f^2_{ijk}(x))$$

into its source label $f^1_{ijk}(x) \in G$ and its morphism label $f^2_{ijk}(x) \in H$.

Identifying $g_{ij} = \phi_{ij}$ this is the gerbe transition law (2.14). $\Box$

Note that the assumption that the base 2-space is simple is crucial for this argument. For more general base 2-spaces the matching condition would instead read

$$t(f^2_{ijk})t\big(g^2_{ik}\big)\, g^1_{ik} = t\big(g^2_{ij}\big)\, g^1_{ij} t\big(g^2_{jk}\big)\, g^1_{jk}\,. \tag{2.26}$$

Only when the simplicity of the base 2-space forces all g^2_{ij} to take values in $\ker(t)$ does this reduce to the transition law for a nonabelian gerbe. But 2-bundles of course exist also for the more general case.

The coherence law for the 2-transition. The natural transformation f which weakens the ordinary transition law $g_{ij}g_{jk} = g_{ik}$ has to satisfy a coherence law which makes its application on multiple products $g_{ij}g_{jk}g_{kl}$ well defined.

Note that first of all the result (2.1) implies a certain relation among the f_{ijk}: By using the relation $g^1_{ij}g^1_{jk} = t(f_{ijk})\, g^1_{ik}$ in the expression $g_{ij}g_{jk}g_{kl}$ in two different ways one obtains

$$t(f_{ijk})\, t(f_{ikl}) = g_{ij} t(f_{jkl})\, g^{-1}_{ij} t(f_{ijl})\ .$$

This equation implies that

$$f^{-1}_{ikl}\, f^{-1}_{ijk} \alpha(g_{ij})(f_{jkl})\, f_{ijl} = \lambda_{ijkl} \tag{2.27}$$

with

$$\lambda_{ijkl} \colon U^1_{ijkl} \to \ker(t) \subset H\,.$$

This is the gerbe transition law (2.18). The function λ_{ijkl} is the 'twist' 0-form (2.13).

From the perspective of 2-bundles the twist can be understood as coming from a nontrivial natural transformation between 2-maps from $U^{[4]}$ to $U^{[2]}$:

First assume that the natural transformation

$$\begin{array}{c} U^{[4]} \overset{j_{023}\circ j_{02}}{\longrightarrow} U^{[2]} \\ \overset{\omega_{03}}{\Rightarrow} \\ U^{[4]} \overset{j_{013}\circ j_{02}}{\longrightarrow} U^{[2]}\,. \end{array} \tag{2.28}$$

is trivial, which means that sending a based loop $\gamma_{(x,i,j,k,l)}$ in $(U^{[4]})^2$ first to the based loop $\gamma_{(x,i,k,l)}$ in $(U^{[3]})^2$ and then to $\gamma_{(x,i,l)}$ in $(U^{[2]})^2$ yields the same result as first sending it to $\gamma_{(x,i,j,l)}$ in $(U^{[3]})^2$ and then to $\gamma_{(x,i,l)}$ in $(U^{[2]})^2$.

Using (2.24) we have

$$
\begin{aligned}
&(g^2_{ij}\cdot g^2_{jk})\cdot g^2_{kl} = g^2_{ij}\cdot(g^2_{jk}\cdot g^2_{kl})\\
\overset{(2.24)}{\Leftrightarrow}\;& ((f_{ijk})^r\circ g^2_{ik}\circ f_{ijk})\cdot\Big(1_{g^1_{kl}}\circ g^2_{kl}\circ 1_{g^1_{kl}}\Big) = \Big(1_{g^1_{ij}}\circ g^2_{ij}\circ 1_{g^1_{ij}}\Big)\cdot((f_{jkl})^r\circ g^2_{jl}\circ f_{jkl})\\
\overset{(2.2)}{\Leftrightarrow}\;& \Big((f_{ijk})^r\cdot 1_{g^1_{kl}}\Big)\circ(g^2_{ik}\cdot g^2_{kl})\circ\Big(f_{ijk}\cdot 1_{g^1_{kl}}\Big) = \Big(1_{g^1_{ij}}\cdot(f_{jkl})^r\Big)\circ(g^2_{ij}\cdot g^2_{jl})\circ\Big(1_{g^1_{ij}}\cdot f_{jkl}\Big)\\
\overset{(2.24)}{\Leftrightarrow}\;& \Big((f_{ijk})^r\cdot 1_{g^1_{kl}}\Big)\circ((f_{ikl})^r\circ g^2_{il}\circ f_{ikl})\circ\Big(f_{ijk}\cdot 1_{g^1_{kl}}\Big) = \Big(1_{g^1_{ij}}\cdot(f_{jkl})^r\Big)\circ((f_{ijl})^r\circ g^2_{il}\circ f^2_{ijl})\circ\Big(1_{g^1_{ij}}\cdot f_{jkl}\Big)\,.
\end{aligned}
\tag{2.29}
$$

This has the form

$$
A^r\circ g^2_{il}\circ A \,=\, B^r\circ g^2_{il}\circ B
$$

with

$$
\begin{aligned}
A &= f_{ikl}\circ(f_{ijk}\cdot 1_{g^1_{kl}})\\
B &= f_{ijl}\circ(1_{g^1_{ij}}\cdot f_{jkl})\,.
\end{aligned}
\tag{2.30}
$$

If we identify both 'conjugations' we obtain

$$
A = B \;\Leftrightarrow\; (f^2_{ikl})^{-1}(f^2_{ijk})^{-1}\alpha\big(g^1_{ij}\big)\big(f^2_{jkl}\big)\, f^2_{ijl} = 1\,. \tag{2.31}
$$

This reproduces (2.27) without the twist.

Now generalize to nontrivial natural transformations (2.28). This implies the existence of a function

$$
(U^{[4]})^1 \xrightarrow{\ell} (U^{[2]})^2
$$

that assigns loops based in double overlaps to points in quadruple overlaps. Applying the 'transition function' g to these loops implies that

$$
g^2(\ell)\circ g^2(j_{013}\circ j_{02}) \,=\, g^2(j_{023}\circ j_{02})\circ g^2(\ell)\ . \tag{2.32}
$$

The 2-group element $g^2(\ell)$ is specified by a function

$$
(U^{[4]})^1 \xrightarrow{\lambda} \ker(t)\subset H
$$

as

$$
\begin{aligned}
(U^{[4]})^1 &\xrightarrow{\ell\circ g} \mathcal{G}^2\\
(x,i,j,k,l) &\;\mapsto\; (g^1_{il},\lambda_{ijkl}(x))\,.
\end{aligned}
$$

All this applies to (2.29) by noting that there on the left hand side the g_{il} in general is $g(j_{023}\circ j_{02})$ while that on the right hand is $g(j_{013}\circ j_{02})$.

Hence in the case of nontrivial arrow base space we have to replace the g^2_{il} in the last line on the left with $g^2(j_{013}\circ j_{02})$ and that on the right with $g^2(\ell)\circ g^2(j_{013}\circ j_{02})\circ(g^2(\ell))^{-1}$.

When doing so the 2-group elements A and B of (2.30) become

$$
\begin{aligned}
A &= (g^1_{il},\lambda^{-1}_{ijkl})\circ f_{ikl}\circ(f_{ijk}\cdot 1_{g^1_{kl}})\\
B &= f_{ijl}\circ(1_{g^1_{ij}}\cdot f_{jkl})\,.
\end{aligned}
$$

Equating these generalizes (2.31) to

$$
A = B \;\Leftrightarrow\; (f^2_{ikl})^{-1}(f^2_{ijk})^{-1}\alpha\big(g^1_{ij}\big)\big(f^2_{jkl}\big)\, f^2_{ijl} = \lambda_{ijkl}\,.
$$

Restriction to the case of trivial base 2-space. It is instructive to restrict the above general disucssion to the case where the base 2-space is trivial (def. 2.5):

In that case the 2-transition specifies the following data:

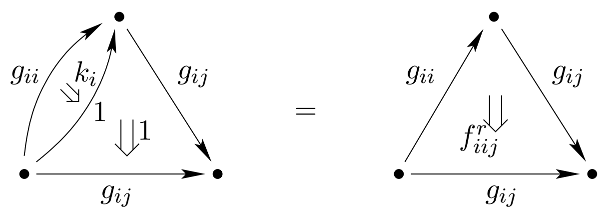
- smooth maps
$$g_{ij} \colon U_i \cap U_j \to \mathcal{G}^1$$
- smooth maps
$$f_{ijk} \colon U_i \cap U_j \cap U_k \to \mathcal{G}^2$$
with
$$f_{ijk}(x) : g_{ik}(x) \to g_{ij}(x)\, g_{jk}(x)$$
- smooth maps
$$k_i : U_i \to \mathcal{G}^2$$
with
$$k_i \colon g_{ii} \to 1 \in \mathcal{G}\,.$$

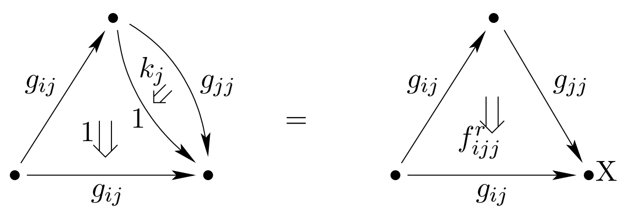
The coherence law (2.27) says that on quadruple intersections $U_i \cap U_j \cap U_k \cap U_l$ the following 2-morphisms in $\mathcal{G}$ are identical:

$$
\begin{tikzcd}
\bullet \arrow[r, "g_{jk}"] & \bullet \arrow[d, "g_{kl}"] \arrow[dl, phantom, "\Downarrow f^r_{ikl}"] & & \bullet \arrow[r, "g_{jk}"] \arrow[dr, "g_{jl}"'] \arrow[d, phantom, "\Downarrow f^r_{ijl}"] & \bullet \arrow[d, "g_{kl}"] \arrow[dl, phantom, "\Downarrow f^r_{jkl}"] \\
\bullet \arrow[u, "g_{ij}"] \arrow[ur, "g_{ik}"] \arrow[r, "g_{il}"'] \arrow[u, phantom, "\Downarrow f^r_{ijk}"'] & \bullet \arrow[r, phantom, "="] & & \bullet \arrow[u, "g_{ij}"] \arrow[r, "g_{il}"', "\lambda_{ijkl}"' below=8pt] & \bullet
\end{tikzcd}
$$

This diagram gives a nice visualization of the different ways to go from the upper arc $g_{ij} g_{jk} g_{kl}$ of the square to the bottom edge g_{il}.

There are also coherence laws for k_i, the **left unit law** and **right unit law**, which express the relation of k to f when two of the indices of the latter coincide:





The freedom of having nontrivial k_i is special to 2-bundles and not known in (nonabelian) gerbe theory. Gerbe cocycles involve Čech cohomology and hence *antisymmetry* in indices $i, j, k, \ldots$ in the sense that group valued functions go into their inverse on an odd permutation of their cover indices.

Whenever we derive nonabelian gerbe cocycles from 2-bundles with 2-connection we will hence have to restrict to $k_i = 1$ for all i.

The above diagrams applied to trivial base 2-spaces. We can also consider nontrivial base 2-spaces, i.e. those with nontrivial base arrow-space. It turns out however that the local description of a 2-bundle coincides with the cocycle data of a nonabelian gerbe only in the limit that the base 2-space morphisms differ 'infinitesimally' from identity morphisms. What this means is explained in the next subsection.

2.2.3 2-Bundles on base 2-Spaces of infinitesimal Loops

The general transition law (2.24) for the arrow part of the transition function g does not seem to have any known counterpart in the theory of nonabelian gerbes (compare the comment on (2.26)). But if we let the morphisms of our base 2-space 'tend to zero' in a certain way, the transition law for the arrow part of the g_{ij} reproduces that of the 'curving transformation 2-form' of a nonabelian gerbe (2.12).

In order to motivate the definition of 'infinitesimal loops' consider some $\mathfrak{h}$-valued 1-form on M and some loop γ_x in M, based at x. When the loop is very small, the holonomy of a along γ is approximately given by

$$W_a[\gamma_x] \approx \exp\left(\int_\Sigma F_a\right) .$$

Where $F_a = da + a \wedge a$ is the curvature of a and Σ is some surface in M with boundary $\partial\Sigma = \gamma$.

For this reason it makes sense to *define* an *infinitesimal* loop based at x to be a 'tangent parallelogram' at x, i.e. an element of the dual space $\Omega^2_{*x}(M)$ of the cotangent space $\Omega^2_x(M)$, which is the space of antisymmetric rank $(2,0)$ tensors (not tensor fields!) over M. The space of all infinitesimal loops naturally has the structure of a 2-space:

Definition 2.16 *The* **free 2-space of infinitesimal loops** *over M, denoted $d\mathrm{L}(M)$, has*

- *point space*

$$(d\mathrm{L}(M))^1 = M$$

- *arrow space*

$$(d\mathrm{L}(M))^2 = \bigcup_{x\in M} \Omega^2_{*x}(M)$$

- *source and target maps*

$$\begin{aligned} d_0, d_1 : (d\mathrm{L}(M))^2 &\to M \\ T_x &\mapsto x \end{aligned}$$

- *composition of arrows*

$$T_x \circ S_x \equiv T_x + S_x \,.$$

Now consider a principal $\mathcal{G}$-2-bundle over M with base 2-space given by $d\mathrm{L}(M)$. The transition functions g^2_{ij} must map these infinitesimal loops to closed 2-morphisms in the structure 2-group $\mathcal{G}$ while respecting composition. In other words, the g^2_{ij} are given by 2-forms

$$d_{ij} \in \Omega^2(U_{ij}, \ker(dt) \subset \mathfrak{h}) \;,$$

such that

$$g^2_{ij}(T_x) \equiv \left(g^1_{ij}(x)\,, \exp(d_{ij}(x)(T_x))\right) \;, \tag{2.33}$$

where on the right the parentheses denote the evaluation of a 2-form $d_{ij}(x)$ on an antisymmetric rank $(2,0)$ tensor T_x.

Since the d_{ij} take values in an abelian subalgebra this action of g^2_{ij} does respect composition of morphisms as it should:

$$\begin{aligned} g^2_{ij}(T_x \circ S_x) &\stackrel{(2.33)}{=} g^2_{ij}(T_x + S_x) \\ &= \left(g^1_{ij}(x)\,, \exp(d_{ij}(T_x))\exp(d_{ij}(S_x))\right) \\ &= g^2_{ij}(T_x) \circ g^2_{ij}(S_x) \;. \end{aligned}$$

Given this definition of spaces of 'infinitesimal loops' we can now prove that for such base 2-spaces the 2-transition of a 2-bundle induces on the arrow-part of g_{ij} a law known from nonabelian gerbe cocycles:

Proposition 2.3 *For a principal $\mathcal{G}$-2-bundle with strict structure 2-group over a base 2-space of free infinitesimal loops (def. 2.16) the 2-transition encodes curving transformation 2-forms d_{ij} (2.12) of a nonabelian gerbe in that the arrow part of the 2-transition (def. 2.15) is equivalent to (2.17) for the special case*

$$\begin{aligned}
& dt(B_i) + F_{A_i} = 0 \\
& d_{ij} \in \Omega^2(U_{ij}, \ker(dt) \subset \mathfrak{h}) \ .
\end{aligned}$$

Proof. The arrow part of the 2-transition law (2.24) is equivalent to

$$\begin{aligned}
(g^1_{ik}, g^2_{ik}) \circ (g^1_{ik}, f^2_{ijk}) &= \left(g^1_{ik}, f^2_{ijk}\right) \circ \left(g^1_{ij} g^1_{jk}, g^2_{ij}\, \alpha\left(g^1_{ij}\right)\left(g^2_{jk}\right)\right) \\
\Leftrightarrow f^2_{ijk} g^2_{ik} (f^2_{ijk})^{-1} &= g^2_{ij}\, \alpha\left(g^1_{ij}\right)\left(g^2_{jk}\right) \ .
\end{aligned}$$

Inserting here the expression (2.33) for g^2_{ij}, the above becomes

$$\begin{aligned}
\exp\Big(f_{ijk}\, d_{ik}(T_x)\ f^{-1}_{ijk}\Big) &= \exp(d_{ij}(T_x)) \exp\left(\alpha\left(g^1_{ij}\right)(d_{jk}(T_x))\right) \\
&= \exp\left(d_{ij}(T_x) + \alpha\left(g^1_{ij}\right)(d_{jk}(T_x))\right) \ .
\end{aligned}$$

Requiring this for all $T_x$implies

$$f_{ijk}\, d_{ik}\, f^{-1}_{ijk} = d_{ij} + \alpha\left(g^1_{ij}\right)(d_{jk}) \ . \tag{2.34}$$

This is indeed the gerbe law (2.17) for the given special case. □

Summary. By internalizing the concept of an ordinary bundle in the 2-category of 2-spaces (which again are categories internalized in Diff, the category of smooth spaces) one obtains a categorified notion of the fiber bundle concept, called 2-bundle, which differs from an ordinary bundle essentially in that what used to be ordinary maps between sets (like the projection map of the bundle) now become (smooth) functors between categories. This adds to the original bundle (at the 'point level' of the 2-space) a dimensional generalization (at the 'arrow level' of the 2-space) of all concepts involved. In addition to providing new 'degrees of freedom' the categorification weakens former notions of equality.

By re-expressing the abstract arrow-theoretic construction of a 2-bundle in terms of concrete local group-valued and algebra-valued p-forms and relations between them, we find a generalization of the ordinary transition laws for such local data in an ordinary bundle. Under certain conditions these generalized transition laws coincide with the cocycle data of nonabelian gerbes.

So far all of this pertained to 2-bundles (and nonabelian gerbes) without a notion of connection. For constructing a categorified connection and hence a notion of nonabelian surface holonomy, it is helpful to first consider ordinary connections on spaces of paths in a manifold. This is the content of the next section.

2.3 Path Space

The space of all 'paths' in a manifold constitutes an infinite-dimensional manifold by itself. In the context of Frechet spaces one can study differential geometry on such infinite dimensional spaces (e.g. [34]). In particular, we study the notion of holonomy of curves in path space. A curve in path space over M maps to a (possibly degenerate) surface in M and hence its path space holonomy gives rise to a notion of surface holonomy in M.

In this section we first discuss basic concepts of differential geometry on path spaces and then apply them to define path space holonomy. Using that, a 2-functor hol from the 2-groupoid of bigons in M (to be defined below) to a strict structure 2-group is defined and shown to be consistent. This functor gives us a notion of *local* 2-holonomy which is used in the subsequent section §2.4 (p.57) to define a *global* 2-holonomy by means of 2-transitions.

Throughout the following, various p-forms taking values in Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are used, where $\mathfrak{g}$ and $\mathfrak{h}$ are equipped with the structure of a differential crossed module $\mathcal{C}$ (def. 2.2).

Elements of a basis of $\mathfrak{g}$ will be denoted by T_a with $a \in (1, \dots, \dim(\mathfrak{g}))$ and those of a basis of $\mathfrak{h}$ by S_a with $a \in (1, \dots, \dim(\mathfrak{h}))$. Arbitrary elements will be expanded as $A = A^a T_a$.

Given a $\mathfrak{g}$-valued 1-form A its **gauge covariant exterior derivative** is

$$\begin{aligned} \mathbf{d}_A \omega &\equiv [\mathbf{d} + A, \omega] \\ &\equiv \mathbf{d}\omega + A^a \wedge d\alpha(T_a)(\omega) \end{aligned}$$

and its **curvature** is

$$\begin{aligned} F_A &\equiv (\mathbf{d} + A)^2 \\ &\equiv \mathbf{d}A + \frac{1}{2} A^a \wedge A^b \, [T_a, T_b] \; . \end{aligned}$$

By a **$\mathcal{C}$-valued (1,2)-form** on a manifold M we shall mean a pair (A, B) with

$$\begin{aligned} A &\in \Omega^1(M, \mathfrak{g}) \\ B &\in \Omega^2(M, \mathfrak{h}) \; . \end{aligned} \tag{2.35}$$

2.3.1 Path Space Differential Calculus

Differential calculus on spaces of *parametrized* paths can relatively easily be handled. We start by establishing some basic facts on parametrized paths and then define the *groupoid of paths* by considering thin homotopy equivalence classes of parametrized paths.

Definition 2.17 *Given a manifold M, the* **based parametrized path space** $\mathcal{P}_s^t(M)$ *over M with source $s \in M$ and target $t \in M$ is the space of smooth maps*

$$\begin{aligned} X : [0,1] &\to M \\ \sigma &\mapsto X(\sigma) \end{aligned} \tag{2.36}$$

which are constant in a neighborhood of $\sigma = 0$ and in a neighborhood of $\sigma = 1$. When source and target coincide

$$\Omega_x(M) \equiv \mathcal{P}_x^x(M)$$

is the **based loop space** *over M based at x.*

The constancy condition at the boundary is known as the property of having **sitting instant**, compare for instance [35]. It serves in def. 2.20 to ensure that the composition of two smooth parametrized paths is again a smooth parametrized path.

We denote a generic path by $X : [0,1] \to M$ or by $\gamma : [0,1] \to M$ depending on whether we want to emphasize that it specifies a point in $\mathcal{P}_s^t(M)$ or a curve in M, respectively.

In the study of differential forms on parametrized path space the following notions play an important role (*cf.* [34], section 2):

Definition 2.18 *1. Given any path space $\mathcal{P}_s^t(M)$ (def. 2.17), the 1-parameter family of maps*

$$\begin{aligned} e_\sigma : \mathcal{P}_s^t(M) &\to M \qquad\qquad (\sigma \in (0,1)) \\ X &\mapsto X(\sigma) \end{aligned}$$

maps each path to its position in M at parameter value σ.

2. Given any differential p-form $\omega \in \Omega^p(M)$ the pullback to $\mathcal{P}_s^t(M)$ by e_σ shall be denoted simply by

$$\omega(\sigma) \equiv e_\sigma^*(\omega) \; .$$

3. With respect to a local coordinate patch on M the differential forms on $\mathcal{P}_s^t(M)$ are generated by the 1-forms $\{dX^\mu(\sigma) \,|\, \mu \in \{1, \dots, \dim(M)\}, \sigma \in (0,1)\}$, which we equivalently write as

$$dX^{(\mu,\sigma)} \equiv dX^\mu(\sigma) \; .$$

Integration over σ will be abbreviated as implicit index contraction, as in

$$f_{(\mu,\sigma)} dX^{(\mu,\sigma)} \equiv \sum_{\mu=1}^{\dim(M)} \int_0^1 d\sigma \; f_\mu(\sigma)\, dX^\mu(\sigma) \; .$$

4. In local coordinates the pullback of any form via e_σ reads

$$\begin{aligned} \omega(\sigma) &= \omega_{\mu_1 \dots \mu_p} \underbrace{\frac{\partial X^{(\mu_1,\sigma)}}{\partial X^{(\nu_1,\rho_1)}}}_{=\delta^{\mu_1}_{\nu_1}\delta(\sigma,\rho_1)} \cdots \underbrace{\frac{\partial X^{(\mu_p,\sigma)}}{\partial X^{(\nu_p,\rho_p)}}}_{=\delta^{\mu_p}_{\nu_p}\delta(\sigma-\rho_p)} (X(\sigma))\; dX^{(\nu_1,\rho_1)} \wedge \cdots \wedge dX^{(\nu_p,\rho_p)} \\ &= \omega_{\mu_1 \dots \mu_p}(X(\sigma))\, dX^{(\mu_1,\sigma)} \wedge \cdots \wedge dX^{(\mu_p,\sigma)} \, . \end{aligned}$$

5. *The vector field*

$$K(X) \equiv \frac{d}{d\sigma} X \\ \equiv X'$$

on path space generates rigid reparameterizations. Its push-forward $e_{\sigma *}(K)(X)$ to M is the tangent to the path X at $X(\sigma)$.

6. *The contraction of $\omega(\sigma)$ with K is denoted by $\iota_K \omega(\sigma)$.*

7. *The exterior differential on path space reads in local coordinates*

$$\mathbf{d} = dX^{(\mu,\sigma)} \wedge \frac{\delta}{\delta X^{(\mu,\sigma)}} . \tag{2.37}$$

A special class of differential forms on path space play a major role:

Definition 2.19 *Given a familiy $\{\omega_i\}_{i=1}^N$ of differential forms on a manifold M with degree*

$$\deg(\omega_i) \equiv p_i + 1$$

one gets a differential form (cf. (def. 2.18))

$$\Omega_{\{\omega_i\},(\alpha,\beta)}(X) \equiv \oint_{X|_\alpha^\beta} (\omega_1, \dots, \omega_n) \equiv \int_{\alpha<\sigma_i<\sigma_{i+1}<\beta} \iota_K \omega_1(\sigma^1) \wedge \cdots \wedge \iota_K \omega_N(\sigma^N)$$

of degree

$$\deg(\Omega_{\{\omega_i\}}) = \sum_{i=1}^N p_i ,$$

on any based parametrized path space $P_s^t(M)$ (def. 2.17).

For $\alpha = 0$ and $\beta = 1$ we write

$$\Omega_{\{\omega_i\}} \equiv \Omega_{\{\omega_i\},(0,1)} .$$

These path space forms are known as **multi integrals** *or* **iterated integrals** *or* **Chen forms** *(cf. [34, 36]).*

It turns out that the exterior derivative on path space maps Chen forms (def. 2.19) to Chen forms in a nice way:

Proposition 2.4 *The action of the path space exterior derivative (2.37) on Chen forms (def. 2.19) is*

$$\mathbf{d} \oint (\omega_1, \cdots, \omega_n) = (\tilde{\mathbf{d}} + \tilde{M}) \oint (\omega_1, \cdots, \omega_n) , \tag{2.38}$$

where

$$\tilde{\mathbf{d}} \oint(\omega_1, \cdots, \omega_n) \equiv -\sum_k (-1)^{\sum\limits_{i<k} p_i} \oint(\omega_1, \cdots, \mathbf{d}\omega_k, \cdots, \omega_n)$$

$$\tilde{M} \oint(\omega_1, \cdots, \omega_n) \equiv -\sum_k (-1)^{\sum\limits_{i<k} p_i} \oint(\omega_1, \cdots, \omega_{k-1} \wedge \omega_k, \cdots, \omega_n)\,,$$

satisfying

$$\begin{aligned} \tilde{\mathbf{d}}^2 &= 0 \\ \tilde{M}^2 &= 0 \\ \left\{\tilde{\mathbf{d}}^2, \tilde{M}^2\right\} &= 0\,. \end{aligned} \tag{2.39}$$

(*cf.* [34, 36])

Before using these facts for the investigation of path space holonomy let us conclude by mentioning

The groupoid of paths.

Definition 2.20 *The* **groupoid of paths** $\mathcal{P}_1(M)$ *in a manifold M is the groupoid for which*

- *objects are points $x \in M$*
- *morphisms with source $s \in M$ and target $t \in M$ are thin homotopy equivalence classes $[\gamma]$ of parametrized paths $\gamma \in \mathcal{P}_s^t(M)$ (def. 2.17)*

$$\begin{tikzcd} x \arrow[r, bend left, "{[\gamma]}"] & y \end{tikzcd}$$

- *composition is given by*

$$\begin{tikzcd} x \arrow[r, bend left, "{[\gamma_1]}"] & y \arrow[r, bend left, "{[\gamma_2]}"] & z \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left, "{[\gamma_1 \circ \gamma_2]}"] & z \end{tikzcd}$$

  where

$$\begin{aligned} \circ \colon \mathcal{P}_x^y(M) \times \mathcal{P}_y^z(M) &\to \mathcal{P}_x^z(M) \\ (\gamma_1, \gamma_2) &\mapsto \gamma_{1,2} \end{aligned}$$

  with

$$\gamma_{1,2}(\sigma) \equiv \begin{cases} \gamma_1(2\sigma) & \text{for } 0 \leq \sigma \leq 1/2 \\ \gamma_2(2\sigma - 1) & \text{for } 1/2 \leq \sigma \leq 1 \end{cases}\,.$$

Note that taking thin homotopy equivalence classes makes this composition asscociative and invertible.

In a similar manner we define the 2-groupoid of bigons in def. 2.30 below.

2.3.2 The Standard Connection 1-Form on Path Space

There are many 1-forms on path space that one could consider as local connection 1-forms in order to define a local holonomy on path space. Here we restrict attention to a special class, to be called the *standard connection 1-forms* (def. 2.23), because, as is shown in §2.3.4 (p.48), these turn out to be the ones which compute local 2-group holonomy. (This same 'standard connection 1-form' can however also be motivated from other points of view, as done in [8, 9].)

Motivation of the standard path space connection 1-form. In order to roughly see how 2-group holonomy gives rise to a connection on path space, consider an iterated horizontal product of 2-group morphisms labeling a row of small 'surface elements' as follows:

$$
\begin{tikzcd}
\bullet \arrow[r, bend left, "g_1"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow h_1"] & \bullet \arrow[r, bend left, "g_2"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow h_2"] & \bullet \arrow[r, bend left, "g_3"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow h_3"] & \bullet \cdots \bullet \arrow[r, bend left, "g_N"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow h_N"] & \bullet \; = \; \bullet \arrow[r, bend left, "g^{\mathrm{tot}}"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow h^{\mathrm{tot}}"] & \bullet
\end{tikzcd}
$$

Here the j-th morphism is supposed to be given by $(g_j, h_j) \in \mathcal{G}$ with $g \in G$ and $h \in H$. By the rules of 2-group multiplication (prop. 2.1) the total horizontal product $(g^{\mathrm{tot}}, h^{\mathrm{tot}})$ is given by

$$
\begin{aligned}
g^{\mathrm{tot}} &= g_1\, g_2\, g_3 \cdots g_N \\
h^{\mathrm{tot}} &= h_1\, \alpha(g_1)(h_2)\; \alpha(g_1 g_2)(h_3) \cdots \alpha(g_1 g_2 g_3 \cdots g_{N-1})(h_N)\;.
\end{aligned}
$$

The products of the g_j can be addressed as a *holonomy* along the upper edges, which, for reasons to become clear shortly, we shall write as

$$
g_1\, g_2 \cdots g_j \equiv (W_{j+1})^{-1}\,.
$$

Now suppose the group elements come from algebra elements $A_j \in \mathfrak{g}$ and $B_j \in \mathfrak{h}$ as

$$
\begin{aligned}
g_j &\equiv \exp(\epsilon A_j) \\
h_j &\equiv \exp\big(\epsilon^2 B_j\big)
\end{aligned}
$$

where

$$
\epsilon \equiv 1/N\,,
$$

then

$$
h^{\mathrm{tot}} = 1 + \epsilon^2 \sum_{j=1}^{N} \alpha\Big(W_j^{-1}\Big)(B_j) + \mathcal{O}\big(\epsilon^4\big)\;.
$$

Using the notation

$$
\begin{aligned}
W_j &\equiv W(1 - j\epsilon, 1) \\
B_j &\equiv B(1 - \epsilon j)
\end{aligned}
$$

we have

$$h^{\text{tot}} = 1 + \epsilon \int\limits_0^1 d\sigma\ \alpha\big(W^{-1}(\sigma,1)\big)(B(\sigma)) + \mathcal{O}\big(\epsilon^3\big)\ .$$

Finally, imagine that the $\mathcal{G}$-labels h_k^{tot} of many such thin horizontal rows of 'surface elements' are composed *vertically*. Each of them comes from algebra elements

$$B_k(\sigma) \equiv B(\sigma, k\epsilon)$$

and holonomies

$$W_k(\sigma,1) \equiv W_{k\epsilon}(\sigma,1)$$

as

$$h_k^{\text{tot}} \equiv 1 + \epsilon \int\limits_0^1 d\sigma\ \alpha\big(W_{k\epsilon}^{-1}(\sigma,1)\big)(B(\sigma,k\epsilon)) + \mathcal{O}\big(\epsilon^3\big)\ .$$

In the limit of vanishing ϵ their total vertical product is

$$\lim_{\epsilon=1/N\to 0} h_0^{\text{tot}} h_\epsilon^{\text{tot}} h_{2\epsilon}^{\text{tot}} \cdots h_1^{\text{tot}} = \mathrm{P}\exp\left(\int\limits_0^1 d\tau\ \mathcal{A}(\tau)\right)$$

for

$$\mathcal{A}(\tau) = \int\limits_0^1 d\sigma\ \alpha\big(W_\tau^{-1}(\sigma,1)\big)(B(\sigma,\tau))\ , \tag{2.40}$$

where P denotes path ordering with respect to τ.

Thinking of each of these vertical rows of surface elements as paths (in the limit $\epsilon \to 0$), this shows roughly how the computation of total 2-group elements from vertical and horizontal products of many 'small' 2-group elements can be reformulated as the holonomy of a connection on path space of the form (2.40).

What is missing in the above discussion is the precise identification of path space differential forms. In the following a path space 1-form having the structure (2.40) is defined and it is shown that indeed its holonomy defines a functor from a 2-groupoid of 'surface elements' ('bigons') to the 2-group $\mathcal{G}$, thus making the above discussion precise.

In order to get there, we first need to deal with some basic issues of holonomies and parallel transport:

Holonomy and parallel transport. In order to set up some notation and conventions and for later references, the following gives a list of well-known definitions and facts that are crucial for the further developments:

Definition 2.21 *Given a path space $\mathcal{P}_s^t(M)$ (def. 2.17) and a $\mathcal{C}$-valued (1,2)-form (A,B) (2.35) on M, the following objects are of interest:*

1. *The* **line holonomy** *of A along a given path X is denoted by*

$$W_A[X]\big(\sigma^1,\sigma^2\big) \equiv \mathrm{P}\exp\left(\int\limits_{X|_{\sigma^1}^{\sigma^2}} A\right) \equiv \sum_{n=0}^{\infty} \oint\limits_{X|_{\sigma^1}^{\sigma^2}} (A^{a_1},\dots,A^{a_n})T_{a_1}\cdots T_{a_n}\,. \tag{2.41}$$

2. *The* **parallel transport** *of elements in $T\in\mathfrak{g}$ and $S\in\mathfrak{h}$ is written*

$$\begin{aligned} W_A[X](\sigma,1)(T(\sigma)) &\equiv T^{W_A[X]}(\sigma) \\ &\equiv W_A^{-1}[X](\sigma,1)\,T(\sigma)\,W_A[X](\sigma,1) \\ &= \sum_{n=0}^{\infty}\oint_{X|_\sigma^1}(-A^{a_1},\cdots,-A^{a_n})\ [T_{a_n},\cdots[T_{a_1},T(\sigma)]\cdots]\ , \\ W_A[X](\sigma,1)(S(\sigma)) &\equiv S^{W_A[X]}(\sigma) \\ &\equiv \alpha\big(W_A^{-1}[X](\sigma,1)\big)(S(\sigma)) \\ &\equiv \sum_{n=0}^{\infty}\oint_{X|_\sigma^1}(-A^{a_1},\cdots,-A^{a_n})\ d\alpha(T_{a_n})\circ\cdots\circ d\alpha(T_{a_1})(S(\sigma))\ . \end{aligned} \tag{2.42}$$

For convenience the dependency $[X]$ on the path X will often be omitted.

Proposition 2.5 *Parallel transport (def. 2.21) has the following properties:*

1. *Let $\sigma_1\le\sigma_2\le\sigma_3$ then*

$$W_A[X](\sigma_1,\sigma_2)\circ W_A[X](\sigma_2,\sigma_3) = W_A[X](\sigma_1,\sigma_3)\ .$$

2. *Conjugation of elements in $\mathfrak{g}$ with parallel tranport of elements in $\mathfrak{h}$ yields*

$$W_A(\sigma,1)\big(d\alpha(T)(\sigma)\big(W_A^{-1}(\sigma,1)(S)\big)\big) = d\alpha\big(T^{W_A}(\sigma)\big)(S)\ . \tag{2.43}$$

3. *Given a G-valued 0-form $g\in\Omega^0(M,G)$ and a path $X\in\mathcal{P}_x^y(M)$ we have*

$$g(x)\,W_A[X](g(y))^{-1} = W_{(gAg^{-1}+g^{-1}\mathbf{d}g)}[X]\,. \tag{2.44}$$

4. *Given a G-valued 0-form $g\in\Omega^0(M,G)$ and a based loop $X\in\mathcal{P}_x^x(M)$ we have*

$$\alpha(\phi(x))(W_A[X](\sigma,1)(S(\sigma))) = W_{A'}[X](\sigma,1)(\alpha(\phi(X(\sigma)))(S(\sigma))) \tag{2.45}$$

with

$$A' \equiv \phi A\phi^{-1} + \phi(d\phi^{-1})\,.$$

Proof.

1. This follows by looking at infinitesimal parallel transport.

2. Integrate up the infinitesimal relation

$$d\alpha\left(1-\epsilon X'\cdot A\right)\left(d\alpha(T)\left(d\alpha\left(1+\epsilon X'\cdot A\right)(S)\right)\right) = d\alpha(T)(S)-\epsilon d\alpha\left(\left[X'\cdot A,T\right]\right)(S)+\mathcal{O}\left(\epsilon^2\right)\ . \tag{2.46}$$

3. Using infinitesimal steps one finds

$$\begin{aligned}
& g(x)\,W_A[X](g(y))^{-1} \\
&= \lim_{\epsilon\to 0} g(x)\,(1+\epsilon X'\cdot A(x))(1+\epsilon X'\cdot A\left(x+\epsilon X'\right))\cdots(1+\epsilon X'\cdot A(y))(g(y))^{-1} \\
&= \lim_{\epsilon\to 0} g(x)\,(1+\epsilon X'\cdot A(x))(g(x+\epsilon))^{-1}g(x+\epsilon)\,(1+\epsilon X'\cdot A\left(x+\epsilon X'\right))\cdots(1+\epsilon X'\cdot A(y))(g(y))^{-1} \\
&= \lim_{\epsilon\to 0} g(x)\,(1+\epsilon X'\cdot A(x))(g(x))^{-1}(1+\epsilon X'\cdot(g(x)\,\mathbf{d}(g(x))^{-1}))g(x+\epsilon)\,(1+\epsilon X'\cdot A\left(x+\epsilon X'\right))\cdots(1+\epsilon X'\cdot A(y))(g(y))^{-1} \\
&= \cdots \\
&= W_{gAg^{-1}+g\mathbf{d}g^{-1}}[X]\,.
\end{aligned} \tag{2.47}$$

4. Again consider infinitesimal steps to obtain

$$\begin{aligned}
& \phi(x)(W_A[X](\sigma_1,1)(S)) \\
&= \lim_{\epsilon\to 0}\phi(x)(W_A(1-\epsilon,1)(W_A(1-2\epsilon,1-\epsilon)(\cdots W_A(\sigma,\sigma+\epsilon)(S\cdots)))) \\
&= \lim_{\epsilon\to 0}\phi(x)\left(W_A(1-\epsilon,1)\left(\phi^{-1}\left(x-\epsilon X'(x)\right)\left(\phi\left(x-\epsilon X'(x)\right)(\cdots)\right)\right)\right) \\
&= \lim_{\epsilon\to 0}\phi(x)\left(W_A(1-\epsilon,1)\left(\phi^{-1}(x)\,(1-\phi(x)\,\epsilon X'\cdot(d\phi^{-1}(x)))\left(\phi\left(x+\epsilon X'(x)\right)(\cdots)\right)\right)\right) \\
&= \ldots \\
&= W_{A'}[X](\sigma,1)(\phi(X(\sigma))(S))\ .
\end{aligned} \tag{2.48}$$

□

As the discussion at the beginning of this section showed, integrals over p-forms pulled back to a path and parallel tranported to some base point play an important role for path space holonomy. Following [36, 9] we introduce special notation to take care of that automatically:

Definition 2.22 *A natural addition to the notation (2.19) for iterated integrals in the presence of a $\mathfrak{g}$-valued 1-form A is the abbreviation*

$$\oint_A(\omega_1,\ldots\omega_N)\equiv\oint\left(\omega_1^{W_A},\ldots\omega_N^{W_A}\right)\ ,$$

where $(\cdot)^{W_A}$ is defined in def 2.21. When Lie algebra indices are displayed on the left they are defined to pertain to the parallel tranported object:

$$\oint_A (\ldots, \omega^a, \ldots) \equiv \oint (\ldots, (\omega^{W_A})^a, \ldots)\,. \tag{2.49}$$

Using this notation first of all the following fact can be conveniently stated, which plays a central role in the analysis of the transition law for the 2-holonomy in §2.4 (p.57):

Proposition 2.6 *The difference in line holonomy (def. 2.21) along a given loop with respect to two different 1-forms A and A' can be expressed as*

$$(W_A[X])^{-1} W_{A'}[X] = \lim_{\epsilon = 1/N \to 0} \left(1 + \epsilon \oint_A (A' - A)\right)\left(1 + \epsilon \oint_{A + \epsilon(A'-A)} (A' - A)\right) \cdots \left(1 + \epsilon \oint_{A' - \epsilon(A'-A)} (A' - A)\right)_X .$$

Proof.

First note that from def. 2.21 it follows that

$$\oint_A (A' - A) = \int_0^1 d\sigma (W_A[X](\sigma, 1))^{-1} \iota_K (A' - A)(\sigma)\, W_A[X](\sigma, 1)\,.$$

This implies that

$$W_A[X]\left(1 + \epsilon \oint_A (A' - A)\right)_X = W_{A+\epsilon(A'-A)}[X] + \mathcal{O}(\epsilon^2)\ .$$

The proposition follows by iterating this. □

Exterior derivative and curvature for Chen forms. The exterior derivative on path space maps Chen forms to Chen forms (prop. 2.4). Since, for reasons explained at the beginning of this section, we shall be interested in Chen forms involving parallel transport (def. 2.22), it is important to know also the particular action of the exterior derivative on these:

Proposition 2.7 *The action of the path space exterior derivative on $\oint_A(\omega)$ is*

$$\mathbf{d} \oint_A (\omega) = -\oint_A (\mathbf{d}_A \omega) - (-1)^{\deg(\omega)} \oint_A (d\alpha(T_a)(\omega)\,, F_A^a)\ . \tag{2.50}$$

(Recall the convention (2.49)).

Proof.

Using the identities

$$\mathbf{d}_A \omega = \mathbf{d}\omega + A^{a_1} \wedge d\alpha(T_{a_1})(\omega) \tag{2.51}$$

and

$$
\begin{aligned}
(\mathbf{d}A^a)\, d\alpha(T_a) + (A^a \wedge A^b)\, d\alpha(T_a) \circ d\alpha(T_b) &= (\mathbf{d}A^a)\, d\alpha(T_a) + (A^a \wedge A^b)\frac{1}{2} d\alpha([T_a, T_b])(S) \\
&= F_A^a\, d\alpha(T_a)\ . \qquad (2.52)
\end{aligned}
$$

the exterior derivative for the case $\deg(C) = \text{odd}$ yields:

$$
\begin{aligned}
& \mathbf{d} \oint (\omega^{W_A}) \\
= \quad & \sum_{n=0}^{\infty} \mathbf{d} \oint (d\alpha(T_{a_n}) \circ \cdots \circ d\alpha(T_{a_1})(\omega)\,, -A^{a_1}, \cdots, -A^{a_n}) \\
\overset{(2.38)}{=} \quad & -\sum_{n=0}^{\infty} \oint (d\alpha(T_{a_n}) \circ \cdots \circ d\alpha(T_{a_1})(\mathbf{d}\omega)\,, -A^{a_1}, \cdots, -A^{a_n}) \\
& - \oint (d\alpha(T_{a_n}) \circ \cdots \circ d\alpha(T_{a_1})(\omega) \wedge (-A^{a_1}), -A^{a_2} \cdots, -A^{a_n}) \\
& - \sum_{n=0}^{\infty} \sum_{1 \leq k \leq n} \oint (d\alpha(T_{a_n}) \circ \cdots \circ d\alpha(T_{a_1})(\omega)\,, iA^{a_1}, \cdots, -A^{a_{k-1}}, \mathbf{d}(-A^{a_k}), -A^{a_{k+1}}, \cdots, -A^{a_n}) \\
& - \sum_{n=0}^{\infty} \sum_{1 < k \leq n} \oint (d\alpha(T_{a_n}) \circ \cdots \circ d\alpha(T_{a_1})(\omega)\,, -A^{a_1}, \cdots, (-A^{a_{k-1}}) \wedge (-A^{a_k}), -A^{a_{k+1}}, \cdots, -A^{a_n}) \\
\overset{(2.51)(2.52)}{=} \quad & -\sum_{n=0}^{\infty} \oint (d\alpha(T_{a_n}) \circ \cdots \circ d\alpha(T_{a_1})(\mathbf{d}_A\omega)\,, -A^{a_1}, \cdots, -A^{a_n}) \\
& + \sum_{n=0}^{\infty} \sum_{1 \leq k \leq n} \oint (d\alpha(T_{a_n}) \circ \cdots \circ d\alpha(T_{a_1})(\omega)\,, -A^{a_1}, \cdots, -A^{a_{k-1}}, iF_A^{a_k}, -A^{a_{k+1}}, \cdots, -A^{a_n}) \\
= \quad & -\oint \left((\mathbf{d}_A\omega)^{W_A}\right) \\
& + \sum_{n=0}^{\infty} \int_0^1 d\sigma_2 \int_0^{\sigma_2} d\sigma_1\; d\alpha(T_{a_n}) \circ \cdots \circ d\alpha(T_{a_1})(d\alpha(T_a)(W_A(\sigma_1, \sigma_2)(\iota_K \omega(\sigma_1)))) \times \\
& \qquad\qquad \times \oint_{\sigma_2}^{1} (F_A^a, -A^{a_1}, \cdots, -A^{a_n}) \\
= \quad & -\oint \left((\mathbf{d}_A\omega)^{W_A}\right) \\
& + \int_0^1 d\sigma_2 \int_0^{\sigma_2} d\sigma_1\; W_A(\sigma_2, 1)(d\alpha(T_a)(W_A(\sigma_1, \sigma_2)(\iota_K \omega(\sigma_1))))\, \iota_K F_A^a(\sigma_2) \\
= \quad & -\oint \left((\mathbf{d}_A\omega)^{W_A}\right) \\
& + \int_0^1 d\sigma_2 \int_0^{\sigma_2} d\sigma_1\; W_A(\sigma_2, 1)\big(d\alpha(T_a)\big(W_A^{-1}(\sigma_2, 1)(W_A(\sigma_1, 1)(\iota_K \omega(\sigma_1)))\big)\big)\, \iota_K F_A^a(\sigma_2)
\end{aligned}
$$

$$\stackrel{(2.43)}{=} \quad -\oint \left((\mathbf{d}_A\omega)^{W_A}\right) + \oint \left(d\alpha(T_a)\left(\omega^{W_A}\right), (F^{W_A})^a\right) .$$

The case $\deg(\omega) = $ even is completely analogous.

□

We have restricted attention here to just a single insertion, i.e. $\oint_A(\omega)$ instead of $\oint_A(\omega_1, \ldots, \omega_n)$, because this is the form that the *standard connection 1-form* has:

Definition 2.23 *Given a $\mathcal{C}$-valued $(1,2)$-form (2.35) the path space 1-form*

$$\Omega^1\left(\mathcal{P}_s^t(M), \mathfrak{h}\right) \ni \mathcal{A}_{(A,B)} \equiv \oint_A (B) .$$

is here called the **standard local connection 1-form on path space***.*

(*cf.* [8, 37, 9])

Given a connection, one wants to know its curvature:

Corollary 2.2 *The curvature of the standard path space 1-form $\mathcal{A}_{(A,B)}$ (def. 2.23) is*

$$\mathcal{F}_{\mathcal{A}} = -\oint_A (\mathbf{d}_A B) - \oint_A (d\alpha(T_a)(B), (F_A + dt(B))^a) . \tag{2.53}$$

Proof. Use 2.23 in 2.7 to write

$$\begin{aligned}
\mathcal{F}_{\mathcal{A}} &\equiv (\mathbf{d}_{\mathcal{A}})^2 \\
&\equiv \mathbf{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \\
&= -\oint_A (\mathbf{d}_A B) - \oint_A (d\alpha(T_a)(B), F_A^a) + \oint_A (B) \wedge \oint_A (B) \\
&\stackrel{(2.3)}{=} -\oint_A (\mathbf{d}_A B) - \oint_A (d\alpha(T_a)(B), (F_A + dt(B))^a) .
\end{aligned} \tag{2.54}$$

□

All this is 'local' in the sense that it makes sense only on some contractible open patch. Suitably generalizing this connection to a globally defined connection is the content of §2.4 (p.57). In the remainder of the present section the local character of the constructions is mostly left implicit.

The form of the curvature of the standard path space connection 1-form suggests to identify the following two objects:

Definition 2.24 *Given a standard path space connection 1-form $\mathcal{A}_{(A,B)}$ (def. 2.23) coming from a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B*

- *the 3-form*

$$H \equiv \mathbf{d}_A B \tag{2.55}$$

is called the **curvature 3-form***,*

- *the 2-form*

$$\tilde{F} \equiv F_A + dt(B) \tag{2.56}$$

is called the **fake curvature 2-form***.*

The term 'fake curvature' has been introduced in [11]. The notation $\tilde{F}$ follows [7]. The curvature 3-form was used in [4].

Using this notation the local path space curvature reads

$$\mathcal{F}_{\mathcal{A}} = -\oint_A (H) - \oint_A \Big(d\alpha(T_a)(B)\,,\tilde{F}^a \Big)\;. \tag{2.57}$$

2.3.3 Path Space Line Holonomy and Gauge Transformations

With the usual tools of differential geometry available for path space (as discussed in §2.3.1 (p.33)) the holonomy on path space is defined as usual:

Definition 2.25 *Given a path space 1-form $\mathcal{A}$ and a curve Σ in path space the* **path space line holonomy** *of $\mathcal{A}$ along Σ is*

$$\mathcal{W}_{\mathcal{A}}(\Sigma) \equiv \mathrm{P}\exp\Big(\int_\Sigma \mathcal{A}\Big)\;.$$

Note that by definition P here indicates path ordering with objects at higher parameter value to the *right* of those with lower parameter value, just as in the definition of ordinary line holonomy in def. (2.21).

Path space line holonomy has a richer set of gauge transformations than holonomy on base space. In fact, ordinary gauge transformations on base space correspond to *constant* ('global') gauge transformations on path space in the following sense:

Proposition 2.8 *Given a path space line holonomy (def. 2.25) coming from a standard path space connection 1-form (def. 2.23) $\mathcal{A}_{(A,B)}$ in a based loop space $\mathcal{P}^x_x(M)$ as well as a G-valued 0-form $\phi \in \Omega^0(M,G)$ we have*

$$\alpha(\phi(x))\Big(\mathcal{W}_{\mathcal{A}_{(A,B)}}(\Sigma)\Big) = \mathcal{W}_{\mathcal{A}_{(A',B')}}(\Sigma)$$

with

$$\begin{aligned} A' &= \phi A \phi^{-1} + \phi(d\phi^{-1}) \\ B' &= \alpha(\phi)(B)\;. \end{aligned}$$

Proof. Write out the path space holonomy in infinitesimal steps and apply (2.45) on each of them. □

The usual notion of gauge transformation is obtained by conjugation:

Definition 2.26 *Given the path space holonomy* $\mathcal{W}_{\mathcal{A}_{(A,B)}}\left(\Sigma|_{X_0}^{X_1}\right)$ *(def. 2.25) of a standard local path space connection 1-form* $\mathcal{A}_{(A,B)}$ *(def. 2.23) along a curve* Σ *in* $\mathcal{P}_s^t(M)$ *with endpaths* X_0 *and* X_1*, an* **infinitesimal path space holonomy gauge transformation** *is the map*

$$\mathcal{W}_{\mathcal{A}_{(A,B)}}\left(\Sigma|_{X_0}^{X_1}\right) \mapsto \left(1 - \epsilon \oint_A (a)\right)_{X_0} \mathcal{W}_{\mathcal{A}_{(A,B)}}\left(\Sigma|_{X_0}^{X_1}\right) \left(1 + \epsilon \oint_A (a)\right)_{X_1} ,$$

for any 1-form

$$a \in \Omega^1(M, \mathfrak{h}) \ .$$

This yields a new sort of gauge transformation in terms of the target space (1,2) form (A, B):

Proposition 2.9 *Infinitesimal path space holonomy gauge transformations (def. 2.26) for the holonomy of a standard path space connection 1-form* $\mathcal{A}_{(A,B)}$ *and arbitrary transformation parameter* a *yields to first order in the parameter* ϵ *the path space holonomy of a transformed standard path space connection 1-form* $\mathcal{A}_{(A',B')}$ *with*

$$\begin{aligned} A' &= A + dt(a) \\ B' &= B - \mathbf{d}_A a \end{aligned} \tag{2.58}$$

if and only if $\mathcal{A}_{(A,B)}$ *is strictly r-flat (def. 2.34). Otherwise the result of the infinitesimal gauge transformation is not (to any non-vanishing order in* ϵ*) the holonomy of any standard path space connection 1-form at all for arbitrary* a*.*

(This was originally considered in [9] for the special case $G = H$, $t = \mathrm{id}$, $\alpha = \mathrm{Ad}$.)
Proof.

As for any holonomy the gauge transformation induces a transformation of the connection 1-form $\mathcal{A} \to \mathcal{A}'$ given by

$$\begin{aligned} \mathcal{A}' &= \left(1 - \epsilon \oint_A (a)\right) (\mathbf{d} + \mathcal{A}) \left(1 + \epsilon \oint_A (a)\right) \\ &= \mathcal{A} + \epsilon \, \mathbf{d}_{\mathcal{A}} \oint_A (a) + \mathcal{O}(\epsilon^2) \ . \end{aligned} \tag{2.59}$$

One finds (using the notation (2.49))

$$\mathcal{A} + \epsilon \, \mathbf{d}_{\mathcal{A}} \oint_A (a) \stackrel{(2.50)}{=} \mathcal{A} - \epsilon \oint_A (\mathbf{d}_A a) + \epsilon \oint_A (d\alpha(T_a)(a) \, , F^a)$$

$$
\begin{aligned}
&\quad +\epsilon\left(\oint_A (B^{a_1}, a^{a_2}) - \oint_A (a^{a_1}, B^{a_2})\right)[S_{a_1}, S_{a_2}] \\
&\overset{(2.3)}{=} \mathcal{A} - \epsilon \oint_A (\mathbf{d}_A a) + \epsilon \oint_A (d\alpha(T_a)(a)\,, F^a) \\
&\quad \mathcal{A} - \epsilon \oint_A (d\alpha(T_a)(B)\,, dt(a)^a) + \epsilon \oint_A (d\alpha(T_a)(a)\,, dt(B)^a) \\
&= \mathcal{A} - \epsilon \oint_A (\mathbf{d}_A a) - \epsilon \oint_A (d\alpha(T_a)(B)\,, dt(a)^a) + \epsilon \oint_A (d\alpha(T_a)(a)\,, (dt(B)+F)^a) \\
&\overset{(2.58)}{=} \oint_{A'} (B') + \epsilon \oint_A (d\alpha(T_a)(a)\,, (dt(B)+F)^a) + \mathcal{O}(\epsilon^2)\ .
\end{aligned}
$$

Since a is by assumption arbitrary, the last line is equal to a standard connection 1-form to order ϵ if and only if $dt(B) + F = 0$. □

The above infinitesimal gauge transformation is easily integrated to a finite gauge transformation:

Definition 2.27 *A* **finite path space holonomy gauge transformation** *is the integration of infinitesimal path space holonomy gauge transformations (def. 2.26), i.e. it is a map for any $a \in \Omega^1(M, \mathfrak{h})$ given by*

$$
\begin{aligned}
& \mathcal{W}_{\mathcal{A}_{(A,B)}}\left(\Sigma|_{X_0}^{X_1}\right) \mapsto \\
& U^{-1}(A,a)\,\mathcal{W}_{\mathcal{A}_{(A,B)}}\left(\Sigma|_{X_0}^{X_1}\right) U(A,a) \\
\equiv & \lim_{\epsilon = 1/N \to 0} \underbrace{\left(1 - \epsilon \oint_{A+dt(a)} (a)\right) \cdots \left(1 - \epsilon \oint_{A+\epsilon dt(a)} (a)\right) \left(1 - \epsilon \oint_A (a)\right)_{X_0}}_{N \text{ factors}} \times \\
& \times \mathcal{W}_{\mathcal{A}_{(A,B)}}\left(\Sigma|_{X_0}^{X_1}\right) \times \\
& \times \underbrace{\left(1 + \epsilon \oint_A (a)\right) \left(1 + \epsilon \oint_{A+\epsilon dt(a)} (a)\right) \cdots \left(1 + \epsilon \oint_{A+dt(a)} (a)\right)_{X_1}}_{N \text{ factors}}\ .
\end{aligned}
$$

Proposition 2.10 *A finite path space holonomy gauge transformation (def. 2.27) of the holonomy of a standard path space connection 1-form $\mathcal{A}_{(A,B)}$ is equivalent to a transformation*

$$
\mathcal{A}_{(A,B)} \mapsto \mathcal{A}_{(A',B')}
$$

where

$$
\begin{aligned}
A &\mapsto A + dt(a) \\
B &\mapsto B - \underbrace{(d_A a + a \wedge a)}_{\equiv k_a}
\end{aligned}
\tag{2.60}
$$

is the transformed $(1,2)$-form (A,B).

Proof.

We have to integrate up (2.58). At each step we have

$$\begin{aligned} A_{(n)} &\equiv A_{(n-1)} + \epsilon\, dt(a) \\ B_{(n)} &\equiv B_{(n-1)} - \epsilon\, da - \epsilon\, A^a_{(n-1)} \wedge d\alpha(T_a)(a) \end{aligned}$$

it follows that

$$\begin{aligned} A' = A_{(N)} &= A + \epsilon N\, dt(a) \\ B' = B_{(N)} &= B - \epsilon N\, da - \epsilon(N-1)A^a \wedge d\alpha(T_a)(a) - \epsilon^2 \frac{N(N-1)}{2} dt(a)^a \wedge d\alpha(T_a)(a)\;, \end{aligned}$$

which in the limit $N = 1/\epsilon \to \infty$ goes to (2.60) (using $\frac{1}{2}dt(a)^a \wedge d\alpha(T_a)(a) \stackrel{(2.3)}{=} a \wedge a$). $\square$

In summary the above yields two different notions of gauge transformations on path space:

1. If the path space in question is a based loop space then according to prop 2.8 a gauge transformation on target space yields an ordinary gauge transformation of the $(1,2)$-form (A, B):

$$\begin{aligned} A &\mapsto \phi A \phi^{-1} + \phi(d\phi^{-1}) \\ B &\mapsto \alpha(\phi)(B)\;. \end{aligned}$$

   We shall call this a **2-gauge transformation of the first kind**.

2. A gauge transformation in path space itself yields, according to prop. 2.10, a transformation

$$\begin{aligned} A &\mapsto A + a \\ B &\mapsto B - (d_A a + a \wedge a)\,. \end{aligned}$$

   We shall call this a **2-gauge transformation of the second kind.**

Recall that according to prop 2.9 this works precisely when (A, B) defines a standard connection 1-form (def. 2.23) on path space for which the 'fake curvature' (def. 2.24) vanishes $\tilde{F} = dt(B) + F_A = 0$.

In the context of loop space these two transformations and the conditions on them were discussed for the special case $G = H$ and $t = \mathrm{id}$, $\alpha = \mathrm{Ad}$ in [9]. In the context of 2-groups and higher lattice gauge theory they were found in section 3.4 of [7]. They also appear in the transition laws for nonabelian gerbes [11, 12, 13], as is discussed in detail in §2.1.4 (p.17). The same transformation for the special case where all groups are abelian is well known from abelian gerbe theory [32] but also for instance from string theory (e.g. section 8.7 of [38]).

With holonomy on path space understood, it is now possible to use the fact that every curve in path space maps to a (possibly degenerate) surface in target space in order to get a notion of (local) surface holonomy. That is the content of the next subsection.

2.3.4 Local 2-Holonomy from local Path Space Holonomy

Just like ordinary holonomy is a functor from the groupoid of paths (def. 2.20) to an ordinary group, 2-holonomy is a 2-functor from some 2-groupoid to a 2-group. This 2-groupoid is roughly that consisting of bounded surfaces in M whose horizontal and vertical composition corresponds to the ordinary gluing of bounded surfaces. This heuristic idea is made precise in the following by constructing $\mathcal{P}_2(M)$, the **2-groupoid of bigons**.

First of all a bigon is a 'surface with two corners'. More precisely:

Definition 2.28 *Given any manifold M a* **parametrized bigon** *in M is a smooth map*

$$\begin{aligned} \Sigma\colon [0,1]^2 &\to M \\ (\sigma,\tau) &\mapsto \Sigma(\sigma,\tau) \end{aligned} \tag{2.61}$$

with

$$\begin{aligned} \Sigma(0,\tau) &= s \in M \\ \Sigma(1,\tau) &= t \in M \end{aligned}$$

for given $s,t \in M$, which is constant in a neighborhood of $\sigma = 0,1$ and independent of τ near $\tau = 0,1$.

Equivalently, a parametrized bigon is path in path space $\mathcal{P}_s^t(M)$ (def. 2.17)

$$\begin{aligned} \Sigma\colon [0,1] &\to \mathcal{P}_s^t(M) \\ \tau &\mapsto \Sigma(\cdot,\tau)\ , \end{aligned}$$

which is constant in a neighborhood of $\tau = 0,1$. We call s the **source vertex** *of the bigon, t the* **target vertex***, $\Sigma(\cdot,0)$ the* **source edge** *and $\Sigma(\cdot,1)$ the* **target edge***.*

As with paths, the parametrization involved here is ultimately not of interest and should be devided out:

Definition 2.29 *An* **unparametrized bigon** *or simply a* **bigon** *is a thin homotopy equivalence class $[\Sigma]$ of parametrized bigons Σ (def. 2.28).*

More in detail, this means (*cf.* for instance [15] p.26 and [2] p.50) that two parametrized bigons $\Sigma_1, \Sigma_2\colon [0,1]^2 \to M$ are taken to be equivalent

$$\Sigma_1 \sim \Sigma_2$$

precisely if there exists a smooth map

$$H\colon [0,1]^3 \to M$$

which takes one bigon smoothly into the other while preserving their boundary, i.e. such that

$$\begin{aligned}
H(\sigma,\tau,0) &= \Sigma_1(\sigma,\tau)\\
H(\sigma,\tau,1) &= \Sigma_2(\sigma,\tau)\\
H(\sigma,0,\nu) &= \Sigma_1(\sigma,0) = \Sigma_2(\sigma,0)\\
H(\sigma,1,\nu) &= \Sigma_1(\sigma,1) = \Sigma_2(\sigma,1)\\
H(0,\tau,\nu) &= \Sigma_1(0,\tau) = \Sigma_2(0,\tau)\\
H(1,\tau,\nu) &= \Sigma_1(1,\tau) = \Sigma_2(1,\tau)\ ,
\end{aligned}$$

but which does so in a degenerate fashion, meaning that

$$\mathrm{rank}(dH)(\sigma,\nu,\tau) < 3$$

for all $\sigma,\tau,\nu \in [0,1]$.

These bigons naturally form a coherent 2-groupoid:

Definition 2.30 *The* **coherent 2-groupoid of bigons** $\mathcal{P}_2(M)$ *in* M *is the groupoid whose*

- *objects are points* $x \in M$
- *morphisms are paths* $\gamma \in \mathcal{P}_x^y(M)$

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma"] & y \end{tikzcd}$$

- *2-morphisms are bigons (def. 2.29) with source edge* γ_1 *and target edge* γ_2

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_2"'] \arrow[r, phantom, "\Downarrow{[\Sigma_1]}"] & y \end{tikzcd}$$

and whose composition operations are defined as

- $$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] & y \arrow[r, bend left, "\gamma_2"] & z \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left, "\gamma_1\circ\gamma_2"] & z \end{tikzcd}$$

- $$\begin{tikzcd} x \arrow[r, bend left=60, "\gamma_1"] \arrow[r, "\gamma_2" description] \arrow[r, bend right=60, "\gamma_3"'] \arrow[r, phantom, shift left=3, "\Downarrow{[\Sigma_1]}"] \arrow[r, phantom, shift right=3, "\Downarrow{[\Sigma_2]}"] & y \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_3"'] \arrow[r, phantom, "\Downarrow{[\Sigma_1\circ\Sigma_2]}"] & y \end{tikzcd}$$

-
$$\begin{array}{ccccc} x & \xrightarrow{\gamma_1} & y & \xrightarrow{\gamma_2} & z \\ & \Downarrow[\Sigma_1] & & \Downarrow[\Sigma_2] & \\ x & \xrightarrow{\gamma_1'} & y & \xrightarrow{\gamma_2'} & z \end{array} \quad = \quad \begin{array}{ccc} x & \xrightarrow{\gamma_1\circ\gamma_2} & z \\ & \Downarrow[\Sigma_1\cdot\Sigma_2] & \\ x & \xrightarrow{\gamma_1'\circ\gamma_2'} & z \end{array}$$

where

$$(\gamma_1 \circ \gamma_2)(\sigma) \equiv \begin{cases} \gamma_1(2\sigma) & \text{for } 0 \le \sigma \le 1/2 \\ \gamma_2(2\sigma - 1) & \text{for } 1/2 \le \sigma \le 1 \end{cases}$$

$$(\Sigma_1 \circ \Sigma_2)(\sigma, \tau) \equiv \begin{cases} \Sigma_1(\sigma, 2\tau) & \text{for } 0 \le \tau \le 1/2 \\ \Sigma_2(\sigma, 2\tau - 1) & \text{for } 1/2 \le \tau \le 1 \end{cases}$$

$$(\Sigma_1 \cdot \Sigma_2)(\sigma, \tau) \equiv \begin{cases} \Sigma_1(2\sigma, \tau) & \text{for } 0 \le \sigma \le 1/2 \\ \Sigma_2(2\sigma - 1, \tau) & \text{for } 1/2 \le \sigma \le 1 \end{cases} .$$

Note that in this definition we did *not* divide out by thin homotopy of parametrized *paths* but only by thin homotopy of parametrized bigons. This implies that the horizontal composition in this 2-groupoid is *not* associative. But one can check that the above indeed is a coherent 2-groupoid where associativity is *weakly* preserved in a *coherent* fashion, as described in [2].

Namely there are degenerate bigons for which $\mathrm{rank}(d\Sigma) \le 1$, whose vertical composition with any other bigon hence has the effect of applying a thin homotopy to that bigon's source or target edges. Therefore associativity of horizontal composition of bigons holds up to vertical composition with such degenerate bigons and hence up to natural isomorphism.

With the 2-groupoid of bigons constructed, 2-holonomy can finally be defined:

Definition 2.31 *Given a manifold M and a strict 2-group $\mathcal{G}$ a* **local strict 2-holonomy** *(or simply* **local 2-holonomy***) is a strict 2-functor*

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$$

from the coherent 2-groupoid of bigons $\mathcal{P}_2(M)$ (def. 2.30) to the strict 2-group $\mathcal{G}$.

(The fact that this functor is strict means that it ignores the parametrization of the bigon's source and target edges. Ultimately one will want to replace the strict 2-group here with a coherent 2-group and the strict 2-functor with some sort of weak 2-functor. See the discussion section §3.2 (p.68).)

We want to construct a local 2-holonomy from a standard path space connection 1-form (def. 2.23). In order to do so we first construct a 'pre-2-holonomy' for any standard path space connection 1-form and then determine under which conditions this actually gives a true 2-holonomy. It turns out that the necessary and sufficient conditions for this is the vanishing of the fake curvature (def. 2.24).

Definition 2.32 *Given a standard path space connection 1-form (def. 2.23) and given any parametrized bigon (def. 2.28)*

$$\Sigma : [0,1]^2 \to M$$

with source edge

$$\gamma_1 \equiv \Sigma(\cdot, 0)$$

and target edge

$$\gamma_2 \equiv \Sigma(\cdot, 1) \ ,$$

the triple

$$(g_1, h, g_2) \in G \times H \times G$$

with

$$\begin{aligned} g_i &\equiv W_A(\gamma_i) \\ h &\equiv \mathcal{W}_{\mathcal{A}}^{-1}(\Sigma(1 - \cdot, \cdot)) \end{aligned} \tag{2.62}$$

is called the **local pre-2-holonomy** *of Σ associated with $\mathcal{A}$.*

(The maybe unexpected inverse and parameter inversion here is just due to the interplay of our conventions on signs and orientations, as will become clear shortly.)

In order for a pre-2-holonomy to give rise to a true 2-holonomy two conditions have to be satisfied:

1. The triple (g_1, h, g_2) has to specify a strict 2-group element. By prop. 2.1 this is the case precisely if $g_2 = t(h)\, g_1$ (2.4).

2. The pre-2-holonomy has to be invariant under thin homotopy in order to be well defined on bigons.

The solution of this is the content of prop. 2.14 below. In order to get there the following considerations are necessary:

Compatibility with strict 2-groups. In order to analyze the first of the above two points consider the behaviour of the pre-2-holonomy under changes of the target edge.

Given a path space $\mathcal{P}_s^t(M)$ and a $\mathfrak{g}$-valued 1-form with line holonomy holonomy $W_A[X]$ on $X \in \mathcal{P}_s^t$ (def. 2.21) the **change in holonomy** of W_A as one changes X is well known to be given by the following:

Proposition 2.11 *Let $\rho : \tau \mapsto X(\tau)$ be the flow generated by the vector field D on $\mathcal{P}_s^t$, then*

$$\left.\frac{d}{d\tau} W_A^{-1}[X(0)] W_A[X(\tau)]\right|_{\tau=0} = -\left(\oint_A (F_A)\right)(D) \ . \tag{2.63}$$

(Note that the right hand side denotes evaluation of the path space 1-form $\oint_A(F_A)$ on the path space vector field D.)

Proof. The proof is standard. The only subtlety is to take care of the various conventions for signs and orientations which give rise to the minus sign in (2.63). $\square$

Proposition 2.12 *For the pre-2-holonomy (def. 2.32) of parametrized bigons Σ associated with the standard connection 1-form $\mathcal{A}_{(A,B)}$ to specify 2-group elements, i.e. for the triples (g_1, h, g_2) to satisfy $g_2 = t(h)\, g_1$, we must have*

$$dt(B) + F_A = 0\,.$$

Proof. According to def. 2.32 the condition $g_2 = t(h)\, g_1$ translates into

$$\begin{aligned} t(h) &= W_A(\gamma_2)\, W_A^{-1}(\gamma_1) \\ &= W_A^{-1}\big(\gamma_2^{-1}\big)\, W_A\big(\gamma_1^{-1}\big)\ . \end{aligned}$$

Now let there be a flow $\tau \mapsto \gamma_\tau$ on $\mathcal{P}_s^t(M)$ generated by a vector field D and choose $\gamma_2^{-1} = \gamma_\tau$ and $\gamma_1^{-1} = \gamma_0$. Then according to prop. 2.11 we have

$$\frac{d}{d\tau} W_A^{-1}\big(\gamma_2^{-1}\big)\, W_A\big(\gamma_1^{-1}\big) = + \left(\oint_A (F_A)\right)_{\gamma_0} (D)\ ,$$

where the plus sign is due to the fact that D here points opposite to the D in prop. 2.11.

Applying the same τ-derivative on the left hand side of (2.64) yields

$$-\left(\oint_A (dt(B))\right)(D) = \left(\oint_A (F_A)\right)(D)\ .$$

(Here the minus sign on the left hand side comes from the fact that we have identified $t(h)$ with the *inverse* path space holonomy $\mathcal{W}_{\mathcal{A}}^{-1}$. This is necessary because the ordinary path space holonomy is path-ordered to the right, while we need $t(h)$ to be path ordered to the left.)

This can be true for all D only if $-dt(B) = F_A$. $\square$

This is nothing but the **nonabelian Stokes theorem**. (Compare for instance [39] and references given there.)

Next it needs to be shown that a pre-2-holonomy with $dt(B) + F_A = 0$ is invariant under thin-homotopy:

Invariance under thin homotopy.

Definition 2.33 *A connection 1-form $\mathcal{A}$ on $\mathcal{P}_s^t(M)$ for all $s,t \in M$ is called* **r-flat** *if its holonomy is invariant under thin homotopy.*

(The notion of r-flatness was introduced by [10], based on [8, 9].)

Proposition 2.13 *The standard path space connection 1-form $\mathcal{A}_{(A,B)}$ (def. 2.23) is r-flat (def. 2.33) precisely if the path space 2-form $\oint_A (d\alpha(T_a)(B)\,,(F_A + dt(B))^a)$ vanishes on all path space vector fields T and $\tilde{T}$ with components $T^{(\mu,\sigma)} = \frac{\partial}{\partial\tau}X_\tau^{(\mu,\sigma)}$ and $\tilde{T}^{(\mu,\sigma)} = a(\tau,\sigma)\,\frac{\partial}{\partial\tau}X_\tau^{(\mu,\sigma)}$ for $\tau \mapsto X_\tau$ any smooth curve in any path space $\mathcal{P}_s^t(M)$ and $(\tau,\sigma) \mapsto a(\tau,\sigma)$ any smooth function, i.e. iff*

$$\oint_A (d\alpha(T_a)(B)\,,(F_A + dt(B))^a)\left(T,\tilde{T}\right) = 0 \tag{2.64}$$

for all T and $\tilde{T}$ of the above form.

Proof. For the special case $G = H$ and $t = \mathrm{id}$, $\alpha = \mathrm{Ad}$ this was proven by [10]. The full proof is a straightforward generalization of this special case:

Let

$$\begin{aligned} (0,1) &\to \mathcal{P}_s^t(M) \\ \tau &\mapsto X_\tau \end{aligned}$$

be any smooth curve in $\mathcal{P}_s^t(M)$ with tangent vector field

$$T(\tau) \equiv \frac{\partial}{\partial\tau}X_\tau^{(\mu,\sigma)}\frac{\delta}{\delta X^{(\mu,\sigma)}}\,.$$

(Recall the notation defined in def. 2.18.)

Flows from this curve through a set of curves which map to thin homotopy equivalent surfaces are generated by vector fields of the form

$$D(\tau) = \int_0^1 d\sigma\,\left(a(\tau,\sigma)\,\frac{\partial}{\partial\tau}X_\tau^\mu(\sigma) + b(\tau,\sigma)\,\frac{\partial}{\partial\sigma}X_\tau^\mu(\sigma)\right)\frac{\delta}{\delta X^\mu(\sigma)}\,.$$

The $\mathcal{A}$-holonomy on any $\mathcal{P}_s^t(M)$ under such a shift vanishes for all choices of $a(\sigma,\tau)$ and $b(\sigma,\tau)$ iff the curvature of $\mathcal{A}$ vanishes when evaluated on these two vector fields:

$$\mathcal{F}_{\mathcal{A}}(T,D) = 0\,.$$

From corollary 2.2 we know that $\mathcal{F} = -\oint(\mathbf{d}_A B) - \oint(d\alpha(T_A)(B)\,,(F_A + dt(B))^a)$. One finds

$$\oint(\mathbf{d}_A B)(T,D) = 0\,,$$

since evaluating this involves evaluating the 3-form $\mathbf{d}_A B$ on three vectors with a degenerate span. For a similar reason

$$\oint(d\alpha(T_A)(B)\,,(F_A + dt(B))^a)\left((\partial_\sigma X^\mu)(\sigma)\,\frac{\delta}{\delta X^\mu(\sigma)},\cdot\right) = 0\,,$$

since this involves evaluating target space 2-forms on two vectors proportional to $\partial_\sigma X^\mu$.

All that remains is hence the condition

$$\oint (d\alpha(T_A)(B)\,,(F_A + dt(B))^a)\Big(T,\tilde{T}\Big) = 0\,.$$

$\square$

From prop 2.12 it is known that the condition for the path space connection to be compatible with the nature of 2-groups is a special case of r-flatness:

Definition 2.34 *An r-flat standard path space connection $\mathcal{A}_{(A,B)}$ (def. 2.23) which solves (2.64) by satisfying*

$$F_A + dt(B) = 0 \tag{2.65}$$

is called **strictly r-flat**.

Now we can finally prove the following:

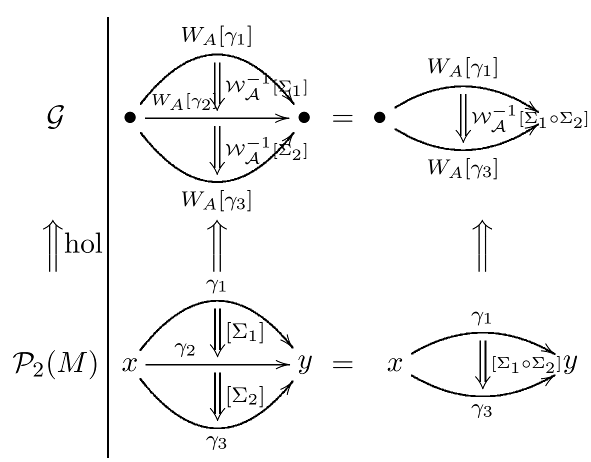
From pre-2-holonomy to true 2-holonomy.

Proposition 2.14 *The pre-2-holonomy (def. 2.32) induces a true local 2-holonomy (def. 2.31) precisely if it comes from a strictly r-flat (def. 2.34) standard path space connection 1-form.*

Proof.

We have already shown that for $dt(B) + F_A = 0$ the pre-2-holonomy indeed maps into a 2-group (prop. 2.12) and that its values are well defined on bigons (prop. 2.13). What remains to be shown is functoriality, i.e. that the pre-2-holonomy respects the composition of bigons and 2-group elements.

First of all it is immediate that composition of paths is respected, due to the properties of ordinary holonomy. Vertical composition of 2-holonomy (being composition of ordinary holonomy in path space) is completely analogous. The fact that pre-2-holonomy involves the *inverse* path space holonomy takes care of the nature of the vertical product in the 2-group, which reverses the order of factors: In the diagram



the top right bigon must be labeled (according to the properties of strict 2-groups described in prop. 2.1) by

$$
\begin{aligned}
(W_A[\gamma_1], \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1]) \circ (W_A[\gamma_2], \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_2]) &= (W_A[\gamma_1], \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_2]\mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1]) \\
&= (W_A[\gamma_1], \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma_1 \circ \Sigma_2]) ,
\end{aligned}
$$

which indeed is the label associated by the hol-functor in the right column of the diagram.

So far we have suppressed in these formulas the reversal (2.62) in the first coordinate of Σ, since it plays no role for the above. This reversal however is essential in order for the hol-functor to respect horizontal composition.

In order to see this it is sufficient to consider *whiskering*, i.e. horizontal composition with identity 2-morphisms.

When whiskering from the left we have

$$
\begin{tikzcd}
\mathcal{G} & \bullet \arrow[r, "W_A[\gamma_1]"] & \bullet \arrow[r, bend left, "W_A[\gamma_2]"] \arrow[r, bend right, "W_A[\gamma_2']"'] \arrow[r, phantom, "\Downarrow \mathcal{W}_{\mathcal{A}}^{-1}[\Sigma]"] & \bullet & = & \bullet \arrow[r, bend left, "W_A[\gamma_1\circ\gamma_2]"] \arrow[r, bend right, "W_A[\gamma_1\circ\gamma_2']"'] \arrow[r, phantom, "\Downarrow \alpha(W_A[\gamma_1])(\mathcal{W}_{\mathcal{A}}^{-1}[\Sigma])"] & \bullet \\
\Uparrow \mathrm{hol} & & \Uparrow & & & \Uparrow & \\
\mathcal{P}_2(M) & x \arrow[r, "\gamma_1"] & y \arrow[r, bend left, "\gamma_2"] \arrow[r, bend right, "\gamma_2'"'] \arrow[r, phantom, "\Downarrow [\Sigma]"] & z & = & x \arrow[r, bend left, "\gamma_1\circ\gamma_2"] \arrow[r, bend right, "\gamma_1\circ\gamma_2'"'] \arrow[r, phantom, "\Downarrow [\Sigma]"] & z
\end{tikzcd}
$$

Evaluating the surface holonomy here involves exponentiating the integrals

$$
\begin{aligned}
\int_{(\gamma_1\circ\gamma_2)^{-1}} d\sigma\; W_A[(\gamma_1 \circ \gamma_2)^{-1}](\sigma,1)(B(\sigma)) &= W_A[\gamma_1^{-1}]\left(\int_{\gamma_2^{-1}} d\sigma\; W_A[\gamma_2^{-1}](\sigma,1)(B(\sigma))\right) \\
&\stackrel{(2.42)}{=} \alpha(W_A[\gamma_1])\left(\int_{\gamma_2^{-1}} d\sigma\; W_A[\gamma_2^{-1}](\sigma,1)(B(\sigma))\right) ,
\end{aligned}
$$

where the contraction with a path space vector tangent to the curve in path space is left implicit. The result in the last line indeed makes the diagram commute. In this computation the path reversal is essential, which of course is related to our convention that parallel transport be to the point with parameter $\sigma = 1$. A simple plausibility argument for this was given at the beginning of §2.3.2 (p.37).

Finally, whiskering to the right is trivial, since we can simply use reparametrization invariance to obtain

$$
\int_{(\gamma_1\circ\gamma_2)^{-1}} d\sigma\; W_A[(\gamma_1 \circ \gamma_2)^{-1}](\sigma,1)(B(\sigma)) = \int_{\gamma_1^{-1}} d\sigma\; W_A[\gamma_1^{-1}](\sigma,1)(B(\sigma)) ,
$$

because for right whiskers the integrand vanishes on γ_2.

Since general horizontal composition is obtained by first whiskering and then composing vertically, this also proves that the hol-functor respects general horizontal composition.

In summary, this shows that a pre-2-holonomy with vanishing fake curvature (def. 2.24) $dt(B) + F_A = 0$ defines a 2-functor $\mathrm{hol}\colon \mathcal{P}_2 \to \mathcal{G}$ and hence a local strict 2-holonomy. $\square$

Before concluding this section it is noteworthy that for vanishing fake curvature the path space curvature 2-form reads (prop. 2.2)

$$\mathcal{F}_{\mathcal{A}} = -\oint_A (H) \,, \tag{2.66}$$

where $H = \mathbf{d}_A B$ is the curvature 3-form (def. 2.24).

Summary. The space of based paths over a given manifold M is an infinite dimensional Frechet manifold on which one can do ordinary differential geometry. A simple semi-heuristic argument shows that an assignment of 2-group elements to 'small surface elements' in M should give rise to a certain local connection 1-form on path space, called the 'standard connection 1-form' which comes from a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B on M.

This can be made precise by defining an appropriate *Chen 1-form* on path space and using its path space line holonomy to assign elements in a strict 2-group $\mathcal{G}$ to (possibly degenerate) surface elements with two corners, called *bigons.* Under obvious composition these bigons naturally form a coherent 2-groupoid $\mathcal{P}_2(M)$ and the assignment of 2-group elements to bigons using the standard path space connection 1-form can be shown to define a functor $\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$ precisely if the 'fake curvature' vanishes, i.e. if the differential forms that the path space connection comes from satisfy $dt(B) + F_A = 0$.

In the next section this result is used to define a global 2-connection in a 2-bundle with strict structure 2-group.

2.4 2-Bundles with 2-Connections

Just like the transition law for the transition functions in a 2-bundle comes from the concept of an ordinary transition *internalized* (§2.1.2 (p.14)) in the 2-category $\mathfrak{C}$ of 2-spaces, where it is called a *2-transition* (def. 2.15), the transition rule for a local 2-connection is obtained by similarly internalizing the ordinary transition of an ordinary connection. A local 2-connection together with its 2-transition then constitues a global 2-connection.

This requires that first of all the ordinary concept of a local connection and its transition in an ordinary bundle be fomulated arrow-theoretically:

An ordinary local connection is a functor hol from the groupoid of paths $\mathcal{P}_1(M)$ (def. 2.20) to the structure group G.

Given some covering U of some base manifold B, this functor in particular gives rise to a map

$$\mathrm{L}U \xrightarrow{W} G\,.$$

from free loops in each patch of the cover to the structure group.

In terms of this map the transition law for the holonomy can be expressed as follows:

Proposition 2.15 *The transition law for the connection 1-form in an ordinary principal bundle with cover U has the arrow-theoretic description:*

$$\begin{aligned}
&\mathrm{L}U^{[2]} \xrightarrow{j_0^{\mathrm{L}}} \mathrm{L}U^{[1]} \xrightarrow{W} G \\
= \\
&\mathrm{L}U^{[2]} \xrightarrow{\overset{\vee}{\mathrm{L}U}{}^{[2]}} \mathrm{L}U^{[2]} \times \mathrm{L}U^{[2]} \xrightarrow{\overset{\vee}{\mathrm{L}U}{}^{[2]} \times \mathrm{L}U^{[2]}} \mathrm{L}U^{[2]} \times \mathrm{L}U^{[2]} \times \mathrm{L}U^{[2]} \longrightarrow \\
&\xrightarrow{d_0 \times j_1^{\mathrm{L}} \times d_0} U^{[2]} \times \mathrm{L}U^{[1]} \times U^{[2]} \xrightarrow{g \times W \times g} G \times G \times G \xrightarrow{m \times s} G \times G \xrightarrow{m} G\,.
\end{aligned} \tag{2.67}$$

(Here the notation follows that of [5]: $\vee$ denotes the diagonal embedding of a space in its second tensor power, m is the multiplication and s the inversion operation in the group. Moreover $\gamma_{(x,i,j)} \overset{j_0^L}{\mapsto} \gamma_{(x,i)}$ and $\gamma_{(x,i,j)} \overset{j_1^L}{\mapsto} \gamma_{(x,j)}$, *cf.* (2.23).)

Proof.

The equality of maps is equivalent to

$$W_i(\gamma_x) = g_{ij}(x)\, W_j(\gamma_x)\, g_{ij}^{-1}(x)\;, \qquad \forall\, \gamma_x \in \mathrm{L}U_{ij}\,, \forall\, i, j\,.$$

By assumption the holonomies W_i come from 1-forms $A_i \in \Omega^1(U_i, \mathfrak{g})$, $W_i = W_{A_i}$ as in (2.41). In terms of these the above is equivalent to

$$W_{A_i}(\gamma_x) = W_{g_{ij}(d+A_j)g_{ij}^{-1}}(\gamma_x)\;, \qquad \forall\, \gamma_x \in \mathrm{L}U_{ij}\,, \forall\, i, j\,.$$

$\square$

2.4.1 2-Transition of 2-Holonomy

Equation (2.67) is internalized in $\mathfrak{C}$ by letting all spaces be 2-spaces and all maps be 2-maps. In particular $\mathrm{L}U^{[2]}$ becomes a 2-loop 2-space:

Definition 2.35 *Given a 2-space S the* **free 2-loop 2-space** $\mathrm{L}S$ *over S is the simple (def. 2.5) 2-space whose point space is S^2 and whose arrow space is the disjoint union of all $\mathcal{P}^{\gamma}_{\gamma}\left(S^2\right)$ (def. 2.17) for all $\gamma \in S^2$ with the obvious source and target maps.*

The local 2-connection functor $\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$ (def. 2.31) gives rise to a 2-map

$$\mathrm{L}U \xrightarrow{W} \mathcal{G}\,. \tag{2.68}$$

with point and arrow part given by (def. 2.32)

$$\begin{aligned} (\mathrm{L}U)^1 &\xrightarrow{W^1} \mathcal{G}^1 \\ \gamma_{(x,i)} &\mapsto W_{A_i}\left(\gamma_{(x,i)}\right)\,, \\[1em] (\mathrm{L}U)^2 &\xrightarrow{W^2} \mathcal{G}^2 \\ \Sigma_{\gamma_{(x,i)}} &\mapsto \left(W_{A_i}\left(\gamma_{(x,i)}\right), \mathcal{W}^{-1}_{\mathcal{A}_i}\left(\Sigma_{\gamma^{-1}_{(x,i)}}\right)\right)\,, \end{aligned} \tag{2.69}$$

in terms of which a global 2-connection is hence defined as follows:

Definition 2.36 *A* **global 2-connection** *in a $\mathcal{G}$-2-bundle with cover 2-space U is a local 2-holonomy* hol *(def. 2.31) on each U_i giving rise to a map (2.68) together with a natural transformation a*

$$\begin{aligned} &\mathrm{L}U^{[2]} \xrightarrow{j_0^{\mathrm{L}}} \mathrm{L}U^{[1]} \xrightarrow{W} G \\ \overset{a}{\Longrightarrow}\; & \\ &\mathrm{L}U^{[2]} \xrightarrow{\mathrm{L}\overset{\vee}{U}{}^{[2]}} \mathrm{L}U^{[2]} \times \mathrm{L}U^{[2]} \xrightarrow{\mathrm{L}\overset{\vee}{U}{}^{[2]} \times \mathrm{L}U^{[2]}} \mathrm{L}U^{[2]} \times \mathrm{L}U^{[2]} \times \mathrm{L}U^{[2]} \longrightarrow \\ &\xrightarrow{d_0 \times j_1^{\mathrm{L}} \times d_0} U^{[2]} \times \mathrm{L}U^{[1]} \times U^{[2]} \xrightarrow{g \times W \times g} \mathcal{G} \times \mathcal{G} \times \mathcal{G} \xrightarrow{m \times s} \mathcal{G} \times \mathcal{G} \xrightarrow{m} \mathcal{G} \end{aligned} \tag{2.70}$$

This somewhat baroque assembly of arrows has the following equivalent description:

Proposition 2.16 *The natural transformation (2.70) defining a global 2-connection in a 2-bundle (def. 2.36) is equivalent to the existence of a map*

$$\begin{aligned} \left(\mathrm{L}U^{[2]}\right)^1 \overset{\text{def. 2.35}}{=} (U^{[2]})^2 &\xrightarrow{\tilde{a}} \mathcal{G}^2 \\ \gamma_{(x,i,j)} &\mapsto \tilde{a}\left(\gamma_{(x,i,j)}\right) \\ &\equiv \left(\tilde{a}^1_{\gamma_{(x,i,j)}}, \tilde{a}^2_{\gamma_{(x,i,j)}}\right) \end{aligned} \tag{2.71}$$

such that

$$\begin{aligned} &\left(W_{A_i}\left(\gamma_{(x,i,j)}\right), \mathcal{W}^{-1}_{\mathcal{A}_i}\left(\Sigma_{\gamma^{-1}_{(x,i,j)}}\right)\right) \circ \left(\tilde{a}^1_{\gamma_{(x,i,j)}}, \tilde{a}^2_{\gamma_{(x,i,j)}}\right) \\ = &\left(\tilde{a}^1_{\gamma_{(x,i,j)}}, \tilde{a}^2_{\gamma_{(x,i,j)}}\right) \circ \left(g_{ij} \cdot \left(W_{A_j}\left(\gamma_{(x,i,j)}\right), \mathcal{W}^{-1}_{\mathcal{A}_j}\left(\Sigma_{\gamma^{-1}_{(x,i,j)}}\right)\right) \cdot g^{-1}_{ij}\right)\,, \qquad \forall\, \gamma_{(x,i,j)} \in U^2_{ij}\,. \end{aligned} \tag{2.72}$$

Here all pairs in brackets denote 2-group elements as in (2.69). "$\circ$" denotes the vertical product in the 2-group and "$\cdot$" the horizontal one.

Proof. This is just the definition of this natural transformation written out. $\square$

We now show what the existence of this map $\tilde{a}$ amounts to in terms of local data. We do this for the case where the arrow part g^2_{ij} of the transition function (def. 2.14) is trivial. According to §2.2.3 (p.30) this implies that in terms of nonabelian gerbe cocycles we now restrict attention to the case where (2.12)

$$d_{ij} = 0\,. \tag{2.73}$$

Proposition 2.17 *A global 2-connection (def. 2.36) on a $\mathcal{G}$-2-bundle with base 2-space being $B = \bigcup_{i\in I}(U^1_i, \mathrm{L}U^1_i)$ and $\mathcal{G}$ a strict automorphism 2-group gives rise to the transition laws (2.15) and (2.16) for the connection 1-form A_i and curving 2-form B_i of a nonabelian gerbe for the special case*

$$\begin{aligned} dt(B_i) + F_{A_i} &= 0\,, \qquad \forall i \\ d_{ij} = 0 &= \beta_{ij}\,, \qquad \forall i,j\,. \end{aligned} \tag{2.74}$$

Proof.

The condition to be studied is the equality (2.72) in prop. 2.16.

In order to avoid awkward notation in the rest of this proof most occurences of γ will be left implicit. Hence first of all (2.72) becomes

$$\left(W_{A_i}, \mathcal{W}^{-1}_{\mathcal{A}_i}\right) \circ \left(\tilde{a}^1_{ij}, \tilde{a}^2_{ij}\right) \;=\; \left(\tilde{a}^1_{ij}, \tilde{a}^2_{ij}\right) \circ \left(g_{ij} \cdot \left(W_{A_j}, \mathcal{W}^{-1}_{\mathcal{A}_j}\right) \cdot (g_{ij})^{-1}\right)\,. \tag{2.75}$$

The source/target matching condition on $\tilde{a}$ is

$$\begin{aligned} \tilde{a}^1_{ij} &= W_{A_i} \\ t\left(\tilde{a}^2_{\gamma(x,i,j)}\right) W_{A_i}[\gamma_{(x,i,j)}] &= g^1_{ij} W_{A_j}[\gamma_{(x,i,j)}](g^1_{ij})^{-1}\,. \end{aligned} \tag{2.76}$$

In order to be able to apply prop. 2.6 to that rewrite this equivalently as

$$W_{A_i}[\gamma^{-1}_{(x,i,j)}]\,t\left((\tilde{a}^2_{\gamma(x,i,j)})^{-1}\right) \;=\; g^1_{ij} W_{A_j}[\gamma^{-1}_{(x,i,j)}](g^1_{ij})^{-1} \tag{2.77}$$

(using $W[\gamma^{-1}] = W^{-1}[\gamma]$).

Hence $t(\tilde{a}^2)$ makes up for the difference between A_i and $g^1_{ij}(\mathbf{d} + A_j)(g^1_{ij})^{-1}$. Denote this difference by

$$a^1_{ij} \equiv g^1_{ij}(\mathbf{d} + A_j)(g^1_{ij})^{-1} - A_i\,. \tag{2.78}$$

Using prop. 2.6 it follows that $t\big(\tilde{a}^2\big)$ can be expressed as

$$t\Big((\tilde{a}^2_{\gamma_{(x,i,j)}})^{-1}\Big) = \lim_{\epsilon=1/N\to 0}\left(1+\epsilon\oint_{A_i}(a^1_{ij})\right)\left(1+\epsilon\oint_{A_i+\epsilon a^1_{ij}}(a^1_{ij})\right)\cdots\left(1+\epsilon\oint_{A_i+(1-\epsilon)a^1_{ij}}(a^1_{ij})\right)_{|\gamma^{-1}_{(x,i,j)}}, \tag{2.79}$$

where the right hand side is evaluated at $\gamma^{-1}_{(x,i,j)}$.

Hence there is an $\mathfrak{h}$-valued 1-form a_{ij} with

$$\begin{aligned} dt(a_{ij}) &\equiv a^1_{ij} \\ &= g^1_{ij}(d+A_j)(g^1_{ij})^{-1} - A_i\,. \end{aligned} \tag{2.80}$$

This is the transition law (2.15) for the connection 1-form of a nonabelian gerbe.

It follows $\tilde{a}^2_{ij}$ itself is given by

$$(\tilde{a}^2_{\gamma_{(x,i,j)}})^{-1} = \lim_{\epsilon=1/N\to 0}\left(1+\epsilon\oint_{A_i}(a_{ij})\right)\left(1+\epsilon\oint_{A_i+\epsilon dt(a_{ij})}(a_{ij})\right)\cdots\left(1+\epsilon\oint_{A_i+(1-\epsilon)dt(a_{ij})}(a_{ij})\right)_{|\gamma^{-1}_{(x,i,j)}}$$

(evaluated at $\gamma^{-1}_{(x,i,j)}$).

Inserting this in (2.72) and using prop. 2.10 one finds

$$\mathcal{W}_{(A_i+dt(a_{ij}),B_i-k_{ij})} = \mathcal{W}_{(g^1_{ij}A_i g^{-1}_{ij}+g^1_{ij}(d(g^1_{ij})^{-1}),\alpha(g^1_{ij})(B_j))}\,. \tag{2.81}$$

This implies

$$B_i = \alpha\big(g^1_{ij}\big)(B_j) + k_{ij} \tag{2.82}$$

with $k_{ij} = \mathbf{d}_A a_{ij} + a_{ij}\wedge a_{ij}$, which is the transition law (2.16) for the curving 2-form of a nonabelian gerbe (for the given special case). $\square$

Note that the natural transformation (2.75) involves a gauge transformation *of the first kind* (2.61) coming from the (horizontal) conjugation with g_{ij} together with a gauge transformation *of the second kind* (2.61) coming from a (vertical) conjugation with $\tilde{a}_{ij}$.

The above natural transformation must be supplemented by a coherence law which ensures its consistency under a chain of compositions translating from U_i to U_j to U_k back to U_i:

Proposition 2.18 *The coherence law of the natural transformation defining a global differentiable 2-connection (def. 2.36) gives the coherence law for the transformators of the connection of a nonabelian gerbe (2.19).*

Proof.

The coherence law ensures consistency of composition of natural transformations. In this case the relevant composition is that of transitions $U_i \to U_j \to U_k \to U_i$.

In terms of the connection 1-form this means, using (2.81)

$$\begin{aligned}
A_i &= g_j^1 A_j (g_j^1)^{-1} + g_j^1(\mathbf{d}(g_j^1)^{-1}) - dt(a_{ij}) \\
&= g_j^1 \left(g_{jk} A_k g_{jk}^{-1} + d_{jk}(\mathbf{d} g_{jk}^{-1}) - a_{jk} \right) (g_j^1)^{-1} + g_j^1(\mathbf{d}(g_j^1)^{-1}) - dt(a_{ij}) \\
&= g_j^1 \left(g_{jk} \left(g_{ki} A_i g_{ki}^{-1} + g_{ki}(\mathbf{d} g_{ki}^{-1}) - dt(a_{ki}) \right) (g_{jk}^1)^{-1} + d_{jk}(\mathbf{d} g_{jk}^{-1}) - dt(a_{jk}) \right) (g_j^1)^{-1} + g_j^1(\mathbf{d}(g_j^1)^{-1}) - dt(a_{ij}) \\
&= (g_{ij}^1 g_{jk}^1 g_{ki}^1) A_i (g_{ij}^1 g_{jk}^1 g_{ki}^1)^{-1} + (g_{ij}^1 g_{jk}^1 g_{ki}^1) \mathbf{d}(g_{ij}^1 g_{jk}^1 g_{ki}^1)^{-1} - dt(a_{ij}) - g_j^1 dt(a_{jk})\, (g_j^1)^{-1} - g_{ij}^1 g_{jk}^1 dt(a_{ki})\, (g_{ij}^1 g_{jk}^1)^{-1} \\
&\overset{(2.25)}{=} A_i + t(f_{ijk}) \left[A_i, t(f_{ijk})^{-1} \right] + t(f_{ijk})\, \mathbf{d} \left(t(f_{ijk})^{-1} \right) - dt(a_{ij}) - g_j^1 dt(a_{jk})\, (g_j^1)^{-1} - t(f_{ijk})\, g_{ik}^1\, dt(a_{ki})\, g_{ik}^1 (t(f_{ijk}))^{-1} \\
&= A_i + dt\Big(f_{ijk} d\alpha(A_i) \left(f_{ijk}^{-1} \right) + f_{ijk} \mathbf{d} f_{ijk}^{-1} - a_{ij} - g_{ij}^1(a_{jk}) - f_{ijk}\, d\alpha\big(g_{ik}^1\big)(a_{ki})\, f_{ijk}^{-1} \Big)\,.
\end{aligned}$$

It follows that

$$f_{ijk} d\alpha(A_i) \left(f_{ijk}^{-1} \right) + f_{ijk} \mathbf{d} f_{ijk}^{-1} - a_{ij} - g_{ij}^1(a_{jk}) - f_{ijk}\, d\alpha\big(g_{ik}^1\big)(a_{ki})\, f_{ijk}^{-1} = -\alpha_{ijk}$$

with $\alpha_{ijk} \in \ker(dt)$. This can be simplified a little: For $j = i$ this equation reduces to

$$a_{ik} + d\alpha\big(g_{ik}^1\big)(a_{ki}) = 0\,. \tag{2.83}$$

Reinserting this result yields

$$f_{ijk} d\alpha(A_i) \left(f_{ijk}^{-1} \right) + f_{ijk} \mathbf{d} f_{ijk}^{-1} - a_{ij} - g_{ij}^1(a_{jk}) + f_{ijk} a_{ki} f_{ijk}^{-1} = -\alpha_{ijk}\,. \tag{2.84}$$

This is indeed the gerbe coherence law (2.19). □

With the transition law for the connection 1-form and 'curving' 2-form understood, the transition law for the curvature 3-form (def. 2.24) follows:

2.4.2 2-Transition of Curvature

Since curvature is the first order term in the holonomy around a small loop, the 2-transition prop. 2.16 of 2-holonomy immediately implies a transition law for the path space connection 1-form $\mathcal{F}_{\mathcal{A}} = -\oint_A (H)$ (2.57) and hence of the curvature 3-form $H = \mathbf{d}_A B$ (def. 2.24).

First of all one notes the following:

Proposition 2.19 *The curvature 3-form (def. 2.24) $H = \mathbf{d}_A B$ transforms* covariantly *under gauge transformations of the first kind (2.61). Moreover, it is* invariant *under gauge transformations of the second kind (2.61) if and only if the fake curvature vanishes.*

Proof.

The covariant transformation under gauge transformations of the first kind follows from simple standard reasoning. The invariance under infinitesimal transformations of the

second kind with $A \to A + \epsilon dt(a)$ and $B \to B - \epsilon \mathbf{d}_A a$ follows from noting the invariance under 'infinitesimal' shifts:

$$\begin{aligned}
H = \mathbf{d}_A B \quad &\to \quad \mathbf{d}_{A+\epsilon dt(a)}(B - \epsilon \mathbf{d}_A a) \\
&= \quad \mathbf{d}_A B - \epsilon \left(\mathbf{d}_A \mathbf{d}_A a - d\alpha(dt(a))(B)\right) + \mathcal{O}\left(\epsilon^2\right) \\
&\stackrel{(2.3)}{=} \quad H - \epsilon \left(d\alpha(F_A)(a) + d\alpha(dt(B))(a)\right) + \mathcal{O}\left(\epsilon^2\right) \\
&\stackrel{(2.56)}{=} \quad H - \epsilon d\alpha\left(\tilde{F}\right)(B) + \mathcal{O}\left(\epsilon^2\right) \\
&\stackrel{\tilde{F}=0}{=} \quad H + \mathcal{O}\left(\epsilon^2\right) .
\end{aligned} \tag{2.85}$$

$\square$

(*cf.* equation (3.43) of [7]).

Note that the invariance of H under transformations of the second kind does *not* imply invariance of the path space curvature 2-form $\mathcal{F}_{\mathcal{A}}$. Instead, this transforms as

$$\mathcal{F}_{\mathcal{A}} = -\oint_A (H) \to -\oint_{A+dt(a)} (H) .$$

Using this expression it is clear that, at least locally, it is always possible to 'gauge away' (by transformations of the second kind) that part of A which takes values in the image $\mathrm{im}(dt)$ of dt. In this sense it is the algebra $\mathfrak{g}/\mathrm{im}(dt)$ that is of relevance for the parallel transport that enters the construction of surface holonomy discussed here. (A related issue is briefly mentioned in the discussion §3.2 (p.68).)

The transition law for $H_i \equiv \mathbf{d}_{A_i} B_i$ is now a simple corollary:

Corollary 2.3 *The local curvature 3-form $H_i = \mathbf{d}_{A_i} B_i$ of the local standard path space connection of a 2-bundle with 2-connection has the transition law*

$$H_i = \alpha\left(g^1_{ij}\right)(H_j)$$

on double intersections U_{ij}.

This is the transition law (2.21) of the curvature 3-form of a nonabelian gerbe for vanishing fake curvature and the special case $d_{ij} = 0$ that we restricted attention to (2.73).

One should note that also the fake curvature (def. 2.24) transforms covariantly and can therefore indeed consistently be chosen to vanish: The transition law for F_{A_i} following from (2.80) is

$$F_{A_i} = g_{ij} F_{A_j} g_{ij}^{-1} - dt(k_{ij})$$

and that of $dt(B)$ following from (2.82)

$$dt(B_i) = g_{ij}\, dt(B_j)\, g_{ij}^{-1} + dt(k_{ij}) \;,$$

so that

$$\tilde{F}_i = g_{ij} \tilde{F}_j g_{ij}^{-1} .$$

Summary. An ordinary local connection gives rise to a local holonomy functor that assigns group elements to paths. These assignments of group elements transform in a well-known way from one patch U_i to another, U_j. In the context of 2-bundles and using the path space technology developed in the previous section, using the concept of a 2-honolomy $\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$ this transition can be *internalized* in $\mathfrak{C}$, the 2-category of 2-spaces, to yield a 2-transition for a 2-connection. When this is worked out in terms of local data the transition law for the connection 1-form and curving 2-form of nonabelian gerbes are obtained, together with their coherence law. From this the gerbe transition law for the curvature 3-form directly follows.

This concludes our analysis of the local description of 2-bundles with 2-connections and the demonstration of the relation to nonabelian gerbes with connection and curving. The main steps of our discussion are summarized in the following section.

3. Summary and Discussion

3.1 Summary of the Constructions and Results

A 2-bundle is the categorification of the notion of an ordinary bundle by internalizing it in the 2-category $\mathfrak{C}$ of 2-spaces, where a 2-space is again a category internalized in Diff (or Diff$_\infty$), the category of finite (infinite) dimensional smooth spaces. The notion of a 2-bundle without connection and the nature of 2-transitions in 2-bundles have been dissussed recently in [5].

We showed first that under certain conditions such a 2-transition in terms of local data reproduce the cocycle description of nonabelian gerbes (without connection and curving). Then we augmented 2-bundles with 2-connections by using holonomy on path space to construct a 2-functor from the 2-groupoid of bigons to the structure 2-group. Using the categorified transition law we globalized the resulting local 2-holonomy.

To begin with, 2-transitions in 2-bundles are characterized by a transition 2-function $U^{[2]} \xrightarrow{g} \mathcal{G}$, a 2-map from double overlaps of the cover 2-space to the structure 2-group. The categorified transition law says that there exists a natural isomorphism between $g_{ij} \cdot g_{jk}$ and g_{ik}.

This means (prop. 2.2) that there exists a map

$$\begin{aligned} (U^{[3]})^1 &\xrightarrow{f} \mathcal{G} \\ (x,i,j,k) &\mapsto f_{ijk}(x) \in \mathcal{G} \\ &\quad = (f^1_{ijk}(x)\,, f^2_{ijk}(x)) \end{aligned}$$

such that

$$(g^1_{ik}, 1) \circ f_{ijk}(x) = f_{ijk}(x) \circ \big((g^1_{ij}(x)\,, 1) \cdot (g^1_{jk}(x)\,, 1) \big)\,. \tag{3.1}$$

Here a superscript 1 denotes the point part of a space or map and a superstript 2 the arrow part. We always denote by '$\circ$' the *vertical* product in the 2-group and by '$\cdot$' the *horizontal* one.

We restrict attention throughout to the case where all morphisms in the base 2-space have coinciding source and target. In this case, at the point level the above says that source and target of this equation must match, which means that

$$t\big(f^2_{ijk}\big)\, g^1_{ik} = g^1_{ij} g^1_{jk}\,.$$

This is the first equation (2.14) of a gerbe cocycle description.

The natural transformation encoded by f can be thought of as modifying the ordinary horizontal product. To emphasize this, equation (3.1) can be equivalently rewritten as

$$(g^1_{ij}(x)\,, 1) \cdot (g^1_{jk}(x)\,, 1) = (g^1_{ik}, f^2_{ijk}(x)\, g^2_{ik}(\gamma_x)\, (f^2_{ijk}(x))^{-1})\,.$$

This 'weakened' version of the ordinary product between transition functions should be consistent, in the sense that

$$(g_{ij} \cdot g_{jk}) \cdot g_{kl} = g_{ij} \cdot (g_{jk} \cdot g_{kl}) \tag{3.2}$$

on double overlaps, where the parentheses indicate in which order we apply the above formula. This is the *coherence law* for the natural transformation encoded by f.

Evaluating this equation yields (2.29) the condition

$$\lambda_{ijkl} = (f^2_{ikl})^{-1}(f^2_{ijk})^{-1}\alpha\big(g^1_{ij}\big)\big(f^2_{jkl}\big)\, f^2_{ijl}\,,$$

for some $\ker(t) \subset H$-valued twist 0-form λ_{ijkl}. This is the gerbe transition law (2.18).

So far this assumes that the arrow part g^2_{ij} of the transition function g is trivial. It can be shown (§2.2.3 (p.30)) that when g^2_{ij} is allowed to take nontrivial values on 'infinitesimal' loops it encodes the 'curving transformation 2-forms' d_{ij} of a nonabelian gerbe (2.12) as well as their transition law (2.17). When discussing the local description of 2-connections on 2-bundles we assume the g^2_{ij} to be trivial in the following.

Now we turn to 2-connections:

Once the notion of 2-holonomy is available, the transition laws for the 2-connection can be dealt with in complete analogy to the above analysis of the transition law for the transition function.

2-holonomy itself is obtained by categorifying ordinary holonomy, which, in a trivial bundle, is a functor $\mathcal{P}_1(M) \to G$. Accordingly, local 2-holonomy should be a 2-functor

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$$

that assigns 2-group elements to bigons in M. Since this functor is supposed to be differentiable there is a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B such that a small bigon with source edge γ and surface Σ is mapped to the 2-group element

$$\approx (\exp\Big(\int_\gamma A\Big)\,, \exp\Big(\int_\Sigma B\Big))\,.$$

This was discussed in [7].

We want to find a connection on path space which computes such a 2-holonomy: Using the rule for horizontal products in a 2-group

$$(g,h)\cdot(g',h') = \big(gg', h\,\alpha(g)\big(h'\big)\big)$$

and iterating it for a long chain of small bigons whose size tends to zero, one finds (§2.3.2 (p.37)) that H-labels of the 2-group elements obtained from the 2-holonomy 2-functor are equivalently computed by the ordinary holonomy of a 1-form $\mathcal{A}$ on path space, given by (see def. 2.23) the integral over a given path of the pullback of B to that path, parallel transported with respect to A to the source vertex of the path.

Precise formulas capturing this simple idea are given in §2.3.1 (p.33). It turns out that the above path space connection defines a 2-functor $\mathcal{P}_2(M) \xrightarrow{\mathrm{hol}} \mathcal{G}$ precisely if the 'fake curvature' vanishes:

$$F_A + dt(B) = 0\,.$$

(This condition is essentially implicit in the very nature of strict 2-groups, as has first been noted in [7].)

The importance of this path space formulation is that it allows to express 2-holonomy differentiably in terms of differential forms and to compute the effect on these differential forms when path space holonomy is conjugated by group elements, as in the transition law for the 2-holonomy:

Denote by $\gamma_{(x,i,j)}$ any element of $U^2_{ij} = LU^1_{ij}$, a closed loop based at x in the double overlap U^1_{ij}. Let $\Sigma_{\gamma_{(x,i,j)}}$ be a bigon with source and target equal to $\gamma_{(x,i,j)}$ (i.e. a closed loop of loops in U^2_{ij}) and denote by

$$\left(W_{A_i}\left(\gamma_{(x,i,j)}\right), \mathcal{W}^{-1}_{\mathcal{A}_i}\left(\Sigma_{\gamma^{-1}_{(x,i,j)}}\right)\right)$$

the 2-group element associated to Σ by the 2-holonomy, where W_{A_i} is the ordinary holonomy of A_i along γ and $\mathcal{W}_{\mathcal{A}_i}$ is the above mentioned path space holonomy of Σ. (The appearance of the inverse is just due to a couple of conventions of signs and orientations that go into this.)

The categorified version of the transition law for this 2-holonomy is (2.72)

$$\begin{aligned}
&\left(W_{A_i}\left(\gamma_{(x,i,j)}\right), \mathcal{W}^{-1}_{\mathcal{A}_i}\left(\Sigma^i_{\gamma^{-1}_{(x,i,j)}}\right)\right) \circ \left(\tilde{a}^1_{\gamma_{(x,i,j)}}, \tilde{a}^2_{\gamma_{(x,i,j)}}\right) \\
&= \left(\tilde{a}^1_{\gamma_{(x,i,j)}}, \tilde{a}^2_{\gamma_{(x,i,j)}}\right) \circ \left(g_{ij} \cdot \left(W_{A_j}\left(\gamma_{(x,i,j)}\right), \mathcal{W}^{-1}_{\mathcal{A}_j}\left(\Sigma^j_{\gamma^{-1}_{(x,i,j)}}\right)\right) \cdot g^{-1}_{ij}\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{L}U^{[2]} &\xrightarrow{\tilde{a}} \mathcal{G} \\
\gamma_{(x,i,j)} &\mapsto \tilde{a}_{ij}(x) \\
&\quad = (\tilde{a}^1_{ij}(x), \tilde{a}^2_{ij}(x))
\end{aligned}$$

encodes the natural transformation.

One should note that beneath the host of symbol decorations this equation has the simple structure of an ordinary 'horizontal' gauge transformation ('of the first kind' (2.61)) by g_{ij} together with an additional 'vertical' gauge transformation ('of the second kind' (2.61)) by $\tilde{a}$.

This is again an equation between 2-maps that splits into its point-part and its arrow-part, from which we can extract the transition laws for the 1-form A_i and the 2-form B_i.

At the point level the target matching condition of this equation says that

$$W_{A_i} t\left((\tilde{a}^2_{ij})^{-1}\right) = g^1_{ij} W_{A_j} (g^1_{ij})^{-1},$$

where we suppress the index $\gamma_{(x,i,j)}$ for convenience. At the arrow level the equation says that

$$\tilde{a}^2_{ij} \mathcal{W}_{\mathcal{A}_i} (\tilde{a}^2_{ij})^{-1} = \alpha(g_{ij})\left(\mathcal{W}_{\mathcal{A}_j}\right).$$

This are essentially our transition laws. It remains to express these equalities between group elements as equalities between the generating algebra elements. This is straightforward but

requires a couple of results on path space differential calculus which are derived in §2.3.1 (p.33). In prop. 2.17 it is shown how using these results the above two equations can be seen to be equivalent to the equations

$$\begin{aligned} A_i + dt(a_{ij}) &= g^1_{ij}(\mathbf{d} + A_j)(g^1_{ij}) \\ B_i &= d\alpha\big(g^1_{ij}\big)(B_j) + k_{ij}\,, \end{aligned}$$

where

$$a_{ij} \in \Omega^1(U_{ij}, \mathfrak{h})$$

is a 1-form taking values in the Lie algebra $\mathfrak{h}$ of H, and

$$k_{ij} \equiv \mathbf{d}_{A_i} a_{ij} + a_{ij} \wedge a_{ij}$$

is its field strength (relative to A_i).

The first of these is precisely the gerbe transition law (2.15) for the connection 1-form A_i. The second is the gerbe transition law (2.16) for the 'curving' 2-form B_i for the case that the gerbe data (2.13) and (2.12) is such that

$$d_{ij} = -\beta_{ij}\,.$$

Next, there is again a coherence law for the natural transformation $\tilde{a}$. Here we need to require that transforming a holonomy from patch i to j to k and then back to i reproduces the original result - up to a twist that arises when when $\gamma_i \neq \gamma_i \mapsto \gamma_j \mapsto \gamma_k \mapsto \gamma_i$.

A straightforward calculation (see prop. 2.18) shows that at the point-space level the coherence law implies that

$$dt\Big(f_{ijk} d\alpha(A_i)\Big(f^{-1}_{ijk}\Big) + f_{ijk}\mathbf{d}f^{-1}_{ijk} - a_{ij} - g^1_{ij}(a_{jk}) + f_{ijk} a_{ki} f^{-1}_{ijk}\Big) = 0\,.$$

This again implies that there is a twist 2-form

$$\alpha_{ijk} \in (\ker(dt)) \times \Omega^2(U_{ijk})$$

taking values in the kernel of dt such that

$$f_{ijk} d\alpha(A_i)\Big(f^{-1}_{ijk}\Big) + f_{ijk}\mathbf{d}f^{-1}_{ijk} - a_{ij} - g^1_{ij}(a_{jk}) + f_{ijk} a_{ki} f^{-1}_{ijk} = -\alpha_{ijk}\,.$$

This is the gerbe transition law (2.18).

Finally, the path space curvature $\mathcal{F}_{\mathcal{A}}$ gives rise to a curvature 3-form $H_i = \mathbf{d}_{A_i} B_i$ (def. 2.24) which (as shown in §2.4.2 (p.61)) has the covariant trasition law

$$H_i = \alpha\big(g^1_{ij}\big)(H_j)\ .$$

This gives us the last two gerbe laws (2.20) and (2.21) subject to the constraints considered here.

In conclusion, a 2-bundle with 2-connection under certain conditions gives rise to a nonabelian gerbe with connection and curving equipped with a notion of nonabelian surface holonomy.

3.2 Discussion and Open Questions

Our central result is that a locally trivializable 2-bundle with simple base 2-space and strict structure 2-group induces the cocylce description of a nonabelian gerbe with connection and curving for vanishing 'fake curvature' and augments the latter with a notion of surface holonomy.

Together with this unifying result we have found a curious dichotomy between generalizations on the 2-bundle side and on the gerbe side. In order to derive the nonabelian gerbe we had to restrict the possible freedom available in 2-bundles by

- restricting the base 2-space to be 'simple',
- restricting the arrow part of the transition functions to be non-trivial only on 'infinitesimal' loops.

In the context of 2-bundles these restrictions appear artificial and should be relaxed. But, as we have discussed, doing so seems to lead to local transition laws that have no counterpart in the literature on nonabelian gerbes.

On the other hand, from the point of view of the study of nonabelian gerbes it is the requirement on the 'fake curvature' to vanish that looks artificial and unsatisfactory.

It is clear, however, that this condition is tightly related to the existence of a notion of 2-holonomy in $\mathcal{G}$-2-bundles for strict $\mathcal{G}$. One important open question is therefore:

- *How does the discussion in this paper generalize when the standard fiber F of the 2-bundle is allowed to be something which is not a strict 2-group?*

The case of most immediate interest is that where the standard fiber is not a strict, but a *coherent* 2-group [2]. 2-bundles with coherent 2-groups are the most general result of the categorification procedure considered here [5]. They are also strongly suggested by the fact that the 2-groupoid of bigons that we use in §2.3.4 (p.48) for construcing 2-holonomy is a coherent 2-groupoid. Consequently 2-holonomy should take values in a coherent 2-group and our discussion should be correspondingly generalized. While allowing coherent structure 2-groups in 2-bundles again drives one out of the realm of the known local description of nonabelian gerbes, it seems possible to generalize this appropriately.

But even with a more general understanding of 2-holonomy currently lacking, the known notion of 2-holonomy (i.e. for strict structure 2-groups) deserves to be better understood. Recall that the path space curvature of the local connection 1-form on path space that we found was

$$\mathcal{F} = \oint_A (H) = \oint_A (\mathbf{d}_A B)$$

which takes values in the *abelian* ideal $\ker(dt) \in \mathfrak{h}$.

When the curvature of an ordinary bundle with connection takes values in a proper ideal of the total algebra the structure group can be reduced to that generated by this ideal. One is therefore led to ask:

- *Is the nonabelian surface holonomy in 2-bundles with strict structure 2-group 're-ducible' in some appropriate sense to ordinary* abelian *surface holonomy?*

The answer to this, either way, seems to be nontrivial due to the fact that even though the path space curvature takes values in an abelian subalgebra it involves nonabelian parallel transport.

Apart from these and similar questions revolving around a better understanding of the general formalism, very interesting questions are related to applications of this formalism to physics. In particular, given that for the first time we now have a notion of global nonabelian surface holonomy, a central question in the context of the topics mentioned in §1.2 (p.7) is:

- *Does this notion of gobal nonabelian surface holonomy correctly describe the coupling of membrane boundaries to stacks of 5-branes?*

  In other words, is this the correct notion of surface holonomy for the 'nonabelian gerbe theories' expected to describe the six-dimensional decompactification of ordinary four-dimensional gauge theory?

  What is the physical interpretation of 'fake curvature' here?

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